is a normed linear space over a field $\mathbb{R}$ with respect to operations addition and standard multiplication which is defined as follows:

(1) $(x_1, x_2, \ldots, x_N) + (y_1, y_2, \ldots, y_N) = (x_1 + y_1, x_2 + y_2, \ldots, x_N + y_N)$ *for all* $(x_1, x_2, \ldots, x_N)$, $(y_1, y_2, \ldots, y_N) \in \mathbb{R}^N$.

(2) $r(x_1, x_2, \ldots, x_N) = (rx_1, rx_2, \ldots, rx_N)$ *for all* $r \in \mathbb{R}$ *and for all* $(x_1, x_2, \ldots, x_N) \in \mathbb{R}^N$.

Proof. H.W. □

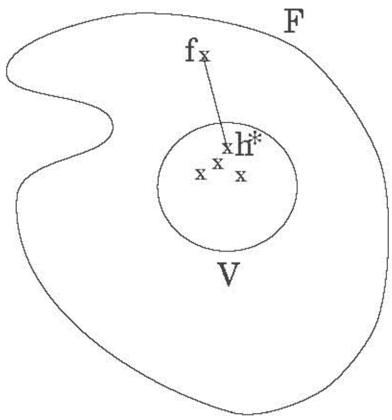
1.2 The Problem of Best Approximation

Let $\mathbb{F}$ be a normed linear space over the field $\mathbb{R}$ and let $\|f\|$ denote the norm of f. Let V be a subset of $\mathbb{F}$, then the general problem of best approximation may be defined in the following terms.

Definition 1.6. *Given a point f and a subset V in a normed linear space $\mathbb{F}$. A best approximation to f from V is an element $h^* \in V$ of minimum distance from f.*
i.e., given $f \in \mathbb{F}$, $f \notin V$, find $h^* \in V$ such that

$$\|f - h^*\| \leqslant \|f - h\| \; \forall \; h \in V.$$

We call h^* a best approximation to f with respect to V and norm $\|\cdot\|$.



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Most of the approximation problems that we consider, and which are of particular interest in practice are of two cases.

(1) Continuous approximation where f and V are in $C[a,b]$.

(2) Discrete approximation where f and V are in $\mathbb{R}^N$.

Remark 1.4. *The Chebyshev norm provides the foundation of much of the approximation theory, the next theorem shows that, if $h \in V$ approximates $f \in \mathbb{F}$ such that $\|E\|_\infty$ is small, where $E = f - h$, then $\|E\|_1$ and $\|E\|_2$ are small too (at least for $b-a$ not too large).*

Theorem 1.3. *For all E in $C[a,b]$ the inequalities*

$$\|E\|_1 \leqslant (b-a)^{\frac{1}{2}}\|E\|_2 \leqslant (b-a)\|E\|_\infty$$

hold.

Proof.

$$\begin{aligned}\|E\|_1 &= \int_a^b |E(x)|dx = \int_a^b |1||E(x)|dx \\ &\leqslant \left[\int_a^b |1|^2 dx\right]^{\frac{1}{2}} \left[\int_a^b |E(x)|^2 dx\right]^{\frac{1}{2}} \quad \text{(By Cauchy-Schwartz inequality)} \\ &\leqslant (b-a)^{\frac{1}{2}}\|E\|_2.\end{aligned}$$

Hence

$$\|E\|_1 \leqslant (b-a)^{\frac{1}{2}}\|E\|_2. \tag{1.3}$$

$$|E(x)| \leqslant \max_{a\leqslant x\leqslant b}|E(x)| = \|E\|_\infty.$$

$$\begin{aligned}\|E\|_2 &= \left[\int_a^b |E(x)|^2 dx\right]^{\frac{1}{2}} \\ &\leqslant \left[\int_a^b \|E\|_\infty^2 dx\right]^{\frac{1}{2}} \\ &\leqslant \|E\|_\infty (b-a)^{\frac{1}{2}}.\end{aligned}$$

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Hence

$$(b-a)^{\frac{1}{2}}\|E\|_2 \leqslant (b-a)\|E\|_\infty. \tag{1.4}$$

from the equations (1.3) and (1.4) we get

$$\|E\|_1 \leqslant (b-a)^{\frac{1}{2}}\|E\|_2 \leqslant (b-a)\|E\|_\infty.$$

□

Remark 1.5. *The converse statement may not be true. i.e., it is not always possible to reduce the $\|E\|_\infty$ by making $\|E\|_1$ or $\|E\|_2$ small, as we see in the following example.*

Example 1.3. *Let $f(x) = 1$, $h(x) = x^\lambda$, λ is a positive parameter, $0 \leqslant x \leqslant 1$.*

Solution. $E = f - h = 1 - x^\lambda$.

$$\|E\|_1 = \int_a^b |E(x)|dx = \int_0^1 |1 - x^\lambda|dx.$$

$$0 \leqslant x \leqslant 1 \quad \Rightarrow \quad 0 \leqslant x^\lambda \leqslant 1 \quad \Rightarrow \quad 0 \geqslant -x^\lambda \geqslant -1 \quad \Rightarrow \quad 0 \leqslant 1 - x^\lambda \leqslant 1.$$

$$|x| = \begin{cases} x, & \text{if } x \geqslant 0; \\ -x, & \text{if } x < 0. \end{cases}$$

Hence

$$\|E\|_1 = \int_0^1 (1 - x^\lambda)dx = x - \frac{x^{\lambda+1}}{\lambda+1}\Big|_0^1 = (1 - \frac{1}{\lambda+1}) - (0-0) = \frac{\lambda}{\lambda+1}.$$

$$\begin{aligned}
\|E\|_2^2 &= \int_a^b |E(x)|^2dx = \int_0^1 |1-x^\lambda|^2dx = \int_0^1 (1-x^\lambda)^2dx = \int_0^1 (1 - 2x^\lambda + x^{2\lambda})dx \\
&= x - 2\frac{x^{\lambda+1}}{\lambda+1} + \frac{x^{2\lambda+1}}{2\lambda+1}\Big|_0^1 = (1 - \frac{2}{\lambda+1} + \frac{1}{2\lambda+1}) - (0-0+0) \\
&= \frac{(\lambda+1)(2\lambda+1) - 2(2\lambda+1) + (\lambda+1)}{(\lambda+1)(2\lambda+1)} = \frac{2\lambda^2 + 3\lambda + 1 - 4\lambda - 2 + \lambda + 1}{(\lambda+1)(2\lambda+1)} \\
&= \frac{2\lambda^2}{(\lambda+1)(2\lambda+1)}.
\end{aligned}$$

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Hence

$$\|E\|_2^2 = \frac{2\lambda^2}{(\lambda+1)(2\lambda+1)}. \quad \Rightarrow \quad \|E\|_2 = \left[\frac{2\lambda^2}{(\lambda+1)(2\lambda+1)}\right]^{\frac{1}{2}}.$$

$$\|E\|_\infty = \max_{a\leqslant x\leqslant b} |E(x)| = \max_{0\leqslant x\leqslant 1} |1 - x^\lambda| = 1.$$

if $\lambda \to 0$, then $\|E\|_1 \to 0$ and $\|E\|_2 \to 0$, but $\|E\|_\infty$ remains 1.

Theorem 1.4. *For all E in $\mathbb{R}^N$ the inequalities*

$$\|E\|_1 \leqslant N^{\frac{1}{2}}\|E\|_2 \leqslant N\|E\|_\infty$$

hold.

Proof. H.W. □

Many question of mathematical interest arise in a natural way from the general best approximation problem (Definition 1.6). For example we may ask the following questions:

(1) Does a best approximation exists?

(2) Is a best approximation unique?

(3) How can a best approximation be characterized?

(4) How can a best approximation be computed?

While we shall refer to these questions, in this lectures the attention will be restricted to the Chebyshev norm as a measure of error.

1.3 Existence

We can investigate an example with regard to question (1).

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