1

BASIC HYPERGEOMETRIC SERIES

1.1 Introduction

Our main objective in this chapter is to present the definitions and notations for hypergeometric and basic hypergeometric series, and to derive the elementary formulas that form the basis for most of the summation, transformation and expansion formulas, basic integrals, and applications to orthogonal polynomials and to other fields that follow in the subsequent chapters. We begin by defining Gauss' ${}_2F_1$ hypergeometric series, the ${}_rF_s$ (generalized) hypergeometric series, and pointing out some of their most important special cases. Next we define Heine's ${}_2\phi_1$ basic hypergeometric series which contains an additional parameter q, called the base, and then give the definition and notations for ${}_r\phi_s$ basic hypergeometric series. Basic hypergeometric series are called q-analogues (basic analogues or q-extensions) of hypergeometric series because an ${}_rF_s$ series can be obtained as the $q \to 1$ limit case of an ${}_r\phi_s$ series.

Since the binomial theorem is at the foundation of most of the summation formulas for hypergeometric series, we then derive a q-analogue of it, called the q-binomial theorem, and use it to derive Heine's q-analogues of Euler's transformation formulas, Jacobi's triple product identity, and summation formulas that are q-analogues of those for hypergeometric series due to Chu and Vandermonde, Gauss, Kummer, Pfaff and Saalschütz, and to Karlsson and Minton. We also introduce q-analogues of the exponential, gamma and beta functions, as well as the concept of a q-integral that allows us to give a q-analogue of Euler's integral representation of a hypergeometric function. Many additional formulas and q-analogues are given in the exercises at the end of the chapter.

1.2 Hypergeometric and basic hypergeometric series

In 1812, Gauss presented to the Royal Society of Sciences at Göttingen his famous paper (Gauss [1813]) in which he considered the infinite series

$$1+\frac{ab}{1\cdot c}z+\frac{a(a+1)b(b+1)}{1\cdot 2\cdot c(c+1)}z^2+\frac{a(a+1)(a+2)b(b+1)(b+2)}{1\cdot 2\cdot 3\cdot c(c+1)(c+2)}z^3+\cdots \quad (1.2.1)$$

as a function of a, b, c, z, where it is assumed that $c \neq 0, -1, -2, \ldots$, so that no zero factors appear in the denominators of the terms of the series. He showed that the series converges absolutely for $|z| < 1$, and for $|z| = 1$ when Re $(c-a-b) > 0$, gave its (contiguous) recurrence relations, and derived his famous formula (see (1.2.11) below) for the sum of this series when $z = 1$ and Re $(c-a-b) > 0$.

Although Gauss used the notation $F(a, b, c, z)$ for his series, it is now customary to use $F(a, b; c; z)$ or either of the notations

$$ {}_2F_1(a,\ b;\ c;\ z), \qquad {}_2F_1\left[\begin{matrix} a, b \\ c \end{matrix}; z\right] $$

for this series (and for its sum when it converges), because these notations separate the numerator parameters a, b from the denominator parameter c and the variable z. In view of Gauss' paper, his series is frequently called *Gauss' series.* However, since the special case $a = 1, b = c$ yields the geometric series

$$ 1 + z + z^2 + z^3 + \cdots, $$

Gauss' series is also called the *(ordinary) hypergeometric series* or the *Gauss hypergeometric series.*

Some important functions which can be expressed by means of Gauss' series are

$$ \begin{aligned} (1+z)^a &= F(-a, b; b; -z), \\ \log(1+z) &= zF(1, 1; 2; -z), \\ \sin^{-1} z &= zF(1/2, 1/2; 3/2; z^2), \\ \tan^{-1} z &= zF(1/2, 1; 3/2; -z^2), \\ e^z &= \lim_{a\to\infty} F(a, b; b; z/a), \end{aligned} \tag{1.2.2} $$

where $|z| < 1$ in the first four formulas. Also expressible by means of Gauss' series are the classical orthogonal polynomials, such as the *Tchebichef polynomials of the first and second kinds*

$$ T_n(x) = F(-n, n; 1/2; (1-x)/2), \tag{1.2.3} $$

$$ U_n(x) = (n+1)F(-n, n+2; 3/2; (1-x)/2), \tag{1.2.4} $$

the *Legendre polynomials*

$$ P_n(x) = F(-n, n+1; 1; (1-x)/2), \tag{1.2.5} $$

the *Gegenbauer (ultraspherical) polynomials*

$$ C_n^\lambda(x) = \frac{(2\lambda)_n}{n!} F(-n, n+2\lambda; \lambda+1/2; (1-x)/2), \tag{1.2.6} $$

and the more general *Jacobi polynomials*

$$ P_n^{(\alpha,\beta)}(x) = \frac{(\alpha+1)_n}{n!} F(-n, n+\alpha+\beta+1;\ \alpha+1; (1-x)/2), \tag{1.2.7} $$

where $n = 0, 1, \ldots$, and $(a)_n$ denotes the *shifted factorial* defined by

$$ (a)_0 = 1,\ (a)_n = a(a+1)\cdots(a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)}, \quad n = 1, 2, \ldots . \tag{1.2.8} $$

Before Gauss, Chu [1303] (see Needham [1959, p. 138], Takács [1973] and Askey [1975, p. 59]) and Vandermonde [1772] had proved the summation formula

$$ F(-n, b; c; 1) = \frac{(c-b)_n}{(c)_n}, \quad n = 0, 1, \ldots, \tag{1.2.9} $$

which is now called *Vandermonde's formula* or the *Chu–Vandermonde formula*, and Euler [1748] had derived several results for hypergeometric series, including his transformation formula

$$F(a,b;c;z) = (1-z)^{c-a-b}\, F(c-a,c-b;c;z), \quad |z|<1. \tag{1.2.10}$$

Formula (1.2.9) is the terminating case $a=-n$ of the summation formula

$$F(a,b;c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \quad \mathrm{Re}(c-a-b)>0, \tag{1.2.11}$$

which Gauss proved in his paper.

Thirty-three years after Gauss' paper, Heine [1846, 1847, 1878] introduced the series

$$1+\frac{(1-q^a)(1-q^b)}{(1-q)(1-q^c)}z+\frac{(1-q^a)(1-q^{a+1})(1-q^b)(1-q^{b+1})}{(1-q)(1-q^2)(1-q^c)(1-q^{c+1})}z^2+\cdots, \tag{1.2.12}$$

where it is assumed that $q\neq 1$, $c\neq 0,-1,-2,\ldots$ and the principal value of each power of q is taken. This series converges absolutely for $|z|<1$ when $|q|<1$ and it tends (at least termwise) to Gauss' series as $q\to 1$, because

$$\lim_{q\to 1}\frac{1-q^a}{1-q} = a. \tag{1.2.13}$$

The series in (1.2.12) is usually called *Heine's series* or, in view of the base q, the *basic hypergeometric series* or *q-hypergeometric series.*

Analogous to Gauss' notation, Heine used the notation $\phi(a,b,c,q,z)$ for his series. However, since one would like to also be able to consider the case when q to the power a, b, or c is replaced by zero, it is now customary to define the *basic hypergeometric series* by

$$\begin{aligned}\phi(a,b;c;q,z) &\equiv {}_2\phi_1(a,b;c;q,z) \equiv {}_2\phi_1\left[\begin{matrix}a,b\\ c\end{matrix};q,z\right]\\ &= \sum_{n=0}^{\infty}\frac{(a;q)_n(b;q)_n}{(q;q)_n(c;q)_n}z^n,\end{aligned} \tag{1.2.14}$$

where

$$(a;q)_n = \begin{cases}1, & n=0,\\ (1-a)(1-aq)\cdots(1-aq^{n-1}), & n=1,2,\ldots,\end{cases} \tag{1.2.15}$$

is the *q-shifted factorial* and it is assumed that $c\neq q^{-m}$ for $m=0,1,\ldots$. Some other notations that have been used in the literature for the product $(a;q)_n$ are $(a)_{q,n}$, $[a]_n$, and even $(a)_n$ when (1.2.8) is not used and the base is not displayed.

Another generalization of Gauss' series is the *(generalized) hypergeometric series* with r numerator parameters $a_1,\ldots,a_r$ and s denominator parameters $b_1,\ldots,b_s$ defined by

$$\begin{aligned}&{}_rF_s(a_1,a_2,\ldots,a_r;\ b_1,\ldots,b_s;z) \equiv {}_rF_s\left[\begin{matrix}a_1,a_2,\ldots,a_r\\ b_1,\ldots,b_s\end{matrix};z\right]\\ &= \sum_{n=0}^{\infty}\frac{(a_1)_n(a_2)_n\cdots(a_r)_n}{n!(b_1)_n\cdots(b_s)_n}z^n.\end{aligned} \tag{1.2.16}$$

Some well-known special cases are the *exponential function*

$$e^z = {}_0F_0(—;—;z), \tag{1.2.17}$$

the *trigonometric functions*

$$\begin{aligned} \sin z &= z\, {}_0F_1(—;3/2;-z^2/4), \\ \cos z &= {}_0F_1(—;1/2;-z^2/4), \end{aligned} \tag{1.2.18}$$

the *Bessel function*

$$J_\alpha(z) = (z/2)^\alpha \, {}_0F_1(—;\alpha+1;-z^2/4)/\Gamma(\alpha+1), \tag{1.2.19}$$

where a dash is used to indicate the absence of either numerator (when $r = 0$) or denominator (when $s = 0$) parameters. Some other well-known special cases are the *Hermite polynomials*

$$H_n(x) = (2x)^n \, {}_2F_0(-n/2,(1-n)/2;—;-x^{-2}), \tag{1.2.20}$$

and the *Laguerre polynomials*

$$L_n^\alpha(x) = \frac{(\alpha+1)_n}{n!} \, {}_1F_1(-n;\alpha+1;x). \tag{1.2.21}$$

Generalizing Heine's series, we shall define an ${}_r\phi_s$ *basic hypergeometric series* by

$$\begin{aligned} {}_r\phi_s(a_1,a_2,\ldots,a_r;b_1,\ldots,b_s;q,z) &\equiv {}_r\phi_s\left[\begin{matrix} a_1,a_2,\ldots,a_r \\ b_1,\ldots,b_s \end{matrix};q,z\right] \\ &= \sum_{n=0}^\infty \frac{(a_1;q)_n(a_2;q)_n\cdots(a_r;q)_n}{(q;q)_n(b_1;q)_n\cdots(b_s;q)_n}\left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-r} z^n \end{aligned} \tag{1.2.22}$$

with $\binom{n}{2} = n(n-1)/2$, where $q \neq 0$ when $r > s+1$.

In (1.2.16) and (1.2.22) it is assumed that the parameters $b_1,\ldots,b_s$ are such that the denominator factors in the terms of the series are never zero. Since

$$(-m)_n = \left(q^{-m};q\right)_n = 0, \quad n = m+1, m+2, \ldots, \tag{1.2.23}$$

an ${}_rF_s$ series terminates if one of its numerator parameters is zero or a negative integer, and an ${}_r\phi_s$ series terminates if one of its numerator parameters is of the form q^{-m} with $m = 0,1,2,\ldots$, and $q \neq 0$. Basic analogues of the classical orthogonal polynomials will be considered in Chapter 7 as well as in the exercises at the ends of the chapters.

Unless stated otherwise, when dealing with nonterminating basic hypergeometric series we shall assume that $|q| < 1$ and that the parameters and variables are such that the series converges absolutely. Note that if $|q| > 1$, then we can perform an inversion with respect to the base by setting $p = q^{-1}$ and using the identity

$$(a;q)_n = (a^{-1};p)_n(-a)^n p^{-\binom{n}{2}} \tag{1.2.24}$$

to convert the series (1.2.22) to a similar series in base p with $|p| < 1$ (see Ex. 1.4(i)). The inverted series will have a finite radius of convergence if the original series does.

Observe that if we denote the terms of the series (1.2.16) and (1.2.22) which contain z^n by u_n and v_n, respectively, then

$$\frac{u_{n+1}}{u_n} = \frac{(a_1+n)(a_2+n)\cdots(a_r+n)}{(1+n)(b_1+n)\cdots(b_s+n)}\, z \tag{1.2.25}$$

is a rational function of n, and

$$\frac{v_{n+1}}{v_n} = \frac{(1-a_1q^n)(1-a_2q^n)\cdots(1-a_rq^n)}{(1-q^{n+1})(1-b_1q^n)\cdots(1-b_sq^n)}\, (-q^n)^{1+s-r}\, z \tag{1.2.26}$$

is a rational function of q^n. Conversely, if $\sum_{n=0}^{\infty} u_n$ and $\sum_{n=0}^{\infty} v_n$ are power series with $u_0 = v_0 = 1$ such that u_{n+1}/u_n is a rational function of n and v_{n+1}/v_n is a rational function of q^n, then these series are of the forms (1.2.16) and (1.2.22), respectively.

By the ratio test, the ${}_rF_s$ series converges absolutely for all z if $r \le s$, and for $|z| < 1$ if $r = s+1$. By an extension of the ratio test (Bromwich [1959, p. 241]), it converges absolutely for $|z| = 1$ if $r = s+1$ and Re $[b_1 + \cdots + b_s - (a_1 + \cdots + a_r)] > 0$. If $r > s+1$ and $z \neq 0$ or $r = s+1$ and $|z| > 1$, then this series diverges, unless it terminates.

If $0 < |q| < 1$, the ${}_r\phi_s$ series converges absolutely for all z if $r \le s$ and for $|z| < 1$ if $r = s+1$. This series also converges absolutely if $|q| > 1$ and $|z| < |b_1b_2\cdots b_sq|/|a_1a_2\cdots a_r|$. It diverges for $z \neq 0$ if $0 < |q| < 1$ and $r > s+1$, and if $|q| > 1$ and $|z| > |b_1b_2\cdots b_sq|/|a_1a_2\cdots a_r|$, unless it terminates. As is customary, the ${}_rF_s$ and ${}_r\phi_s$ notations are also used for the sums of these series inside the circle of convergence and for their analytic continuations (called *hypergeometric functions* and *basic hypergeometric functions*, respectively) outside the circle of convergence.

Observe that the series (1.2.22) has the property that if we replace z by z/a_r and let $a_r \to \infty$, then the resulting series is again of the form (1.2.22) with r replaced by $r-1$. Because this is not the case for the ${}_r\phi_s$ series defined without the factors $\left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-r}$ in the books of Bailey [1935] and Slater [1966] and we wish to be able to handle such limit cases, we have chosen to use the series defined in (1.2.22). There is no loss in generality since the Bailey and Slater series can be obtained from the $r = s+1$ case of (1.2.22) by choosing s sufficiently large and setting some of the parameters equal to zero.

An ${}_{r+1}F_r$ series is called *k-balanced* if $b_1 + b_2 + \cdots + b_r = k + a_1 + a_2 + \cdots + a_{r+1}$ and $z = 1$; a 1-balanced series is called *balanced* (or *Saalschützian*). Analogously, an ${}_{r+1}\phi_r$ series is called *k-balanced* if $b_1b_2\cdots b_r = q^k a_1a_2\cdots a_{r+1}$ and $z = q$, and a 1-balanced series is called *balanced* (or *Saalschützian*). We will first encounter balanced series in §1.7, where we derive a summation formula for such a series.

For negative subscripts, the *shifted factorial* and the *q-shifted factorials* are defined by

$$(a)_{-n} = \frac{1}{(a-1)(a-2)\cdots(a-n)} = \frac{1}{(a-n)_n} = \frac{(-1)^n}{(1-a)_n}, \tag{1.2.27}$$

$$(a;q)_{-n}=\frac{1}{(1-aq^{-1})(1-aq^{-2})\cdots(1-aq^{-n})}=\frac{1}{(aq^{-n};q)_n}=\frac{(-q/a)^n q^{\binom{n}{2}}}{(q/a;q)_n}, \tag{1.2.28}$$

where $n=0,1,\ldots$. We also define

$$(a;q)_\infty=\prod_{k=0}^{\infty}(1-aq^k) \tag{1.2.29}$$

for $|q|<1$. Since the infinite product in (1.2.29) diverges when $a\neq 0$ and $|q|\geq 1$, whenever $(a;q)_\infty$ appears in a formula, we shall assume that $|q|<1$. The following easily verified identities will be frequently used in this book:

$$(a;q)_n=\frac{(a;q)_\infty}{(aq^n;q)_\infty}, \tag{1.2.30}$$

$$(a^{-1}q^{1-n};q)_n=(a;q)_n(-a^{-1})^n q^{-\binom{n}{2}}, \tag{1.2.31}$$

$$(a;q)_{n-k}=\frac{(a;q)_n}{(a^{-1}q^{1-n};q)_k}\,(-qa^{-1})^k q^{\binom{k}{2}-nk}, \tag{1.2.32}$$

$$(a;q)_{n+k}=(a;q)_n(aq^n;q)_k, \tag{1.2.33}$$

$$(aq^n;q)_k=\frac{(a;q)_k(aq^k;q)_n}{(a;q)_n}, \tag{1.2.34}$$

$$(aq^k;q)_{n-k}=\frac{(a;q)_n}{(a;q)_k}, \tag{1.2.35}$$

$$(aq^{2k};q)_{n-k}=\frac{(a;q)_n(aq^n;q)_k}{(a;q)_{2k}}, \tag{1.2.36}$$

$$(q^{-n};q)_k=\frac{(q;q)_n}{(q;q)_{n-k}}(-1)^k q^{\binom{k}{2}-nk}, \tag{1.2.37}$$

$$(aq^{-n};q)_k=\frac{(a;q)_k(qa^{-1};q)_n}{(a^{-1}q^{1-k};q)_n}q^{-nk}, \tag{1.2.38}$$

$$(a;q)_{2n}=(a;q^2)_n(aq;q^2)_n, \tag{1.2.39}$$

$$(a^2;q^2)_n=(a;q)_n(-a;q)_n, \tag{1.2.40}$$

where n and k are integers. A more complete list of useful identities is given in Appendix I at the end of the book.

Since products of q-shifted factorials occur so often, to simplify them we shall frequently use the more compact notations

$$(a_1,a_2,\ldots,a_m;q)_n=(a_1;q)_n(a_2;q)_n\cdots(a_m;q)_n, \tag{1.2.41}$$

$$(a_1,a_2,\ldots,a_m;q)_\infty=(a_1;q)_\infty(a_2;q)_\infty\cdots(a_m;q)_\infty. \tag{1.2.42}$$

The ratio $(1-q^a)/(1-q)$ considered in (1.2.13) is called a *q-number* (or *basic number*) and it is denoted by

$$[a]_q = \frac{1-q^a}{1-q}, \quad q \neq 1. \tag{1.2.43}$$

It is also called a q-analogue, q-deformation, q-extension, or a q-generalization of the complex number a. In terms of q-numbers the *q-number factorial* $[n]_q!$ is defined for a nonnegative integer n by

$$[n]_q! = \prod_{k=1}^{n} [k]_q, \tag{1.2.44}$$

and the corresponding *q-number shifted factorial* is defined by

$$[a]_{q;n} = \prod_{k=0}^{n-1} [a+k]_q. \tag{1.2.45}$$

Clearly,

$$\lim_{q\to 1} [n]_q! = n!, \quad \lim_{q\to 1} [a]_q = a, \tag{1.2.46}$$

and

$$[a]_{q;n} = (1-q)^{-n}(q^a;q)_n, \quad \lim_{q\to 1} [a]_{q;n} = (a)_n. \tag{1.2.47}$$

Corresponding to (1.2.41) we can use the compact notation

$$[a_1, a_2, \ldots, a_m]_{q;n} = [a_1]_{q;n}[a_2]_{q;n} \cdots [a_m]_{q;n}. \tag{1.2.48}$$

Since

$$\begin{aligned} &\sum_{n=0}^{\infty} \frac{[a_1, a_2, \ldots, a_r]_{q;n}}{[n]_q![b_1, \ldots, b_s]_{q;n}} \left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-r} z^n \\ &= {}_r\phi_s\left(q^{a_1}, q^{a_2}, \ldots, q^{a_r}; q^{b_1}, \ldots, q^{b_s}; q, z(1-q)^{1+s-r}\right), \end{aligned} \tag{1.2.49}$$

anyone working with q-numbers and the *q-number hypergeometric series* on the left-hand side of (1.2.49) can use the formulas for ${}_r\phi_s$ series in this book that have no zero parameters by replacing the parameters by $q^{\tau\eta}$ powers and applying (1.2.49).

As in Frenkel and Turaev [1995] one can define a *trigonometric number* $[a;\sigma]$ by

$$[a;\sigma] = \frac{\sin(\pi\sigma a)}{\sin(\pi\sigma)} \tag{1.2.50}$$

for noninteger values of σ and view $[a;\sigma]$ as a *trigonometric deformation* of a since $\lim_{\sigma\to 0}[a;\sigma] = a$. The corresponding ${}_rt_s$ *trigonometric hypergeometric series* can be defined by

$$\begin{aligned} &{}_rt_s(a_1, a_2, \ldots, a_r; b_1, \ldots, b_s; \sigma, z) \\ &= \sum_{n=0}^{\infty} \frac{[a_1, a_2, \ldots, a_r; \sigma]_n}{[n;\sigma]![b_1, \ldots, b_s; \sigma]_n} \left[(-1)^n e^{\pi i\sigma\binom{n}{2}}\right]^{1+s-r} z^n, \end{aligned} \tag{1.2.51}$$

where

$$[n;\sigma]! = \prod_{k=1}^{n}[k;\sigma], \quad [a;\sigma]_n = \prod_{k=0}^{n-1}[a+k;\sigma], \tag{1.2.52}$$

and

$$[a_1, a_2, \ldots, a_m;\sigma]_n = [a_1;\sigma]_n[a_2;\sigma]_n\cdots[a_m;\sigma]_n. \tag{1.2.53}$$

From

$$[a;\sigma] = \frac{e^{\pi i\sigma a} - e^{-\pi i\sigma a}}{e^{\pi i\sigma} - e^{-\pi i\sigma}} = \frac{q^{a/2} - q^{-a/2}}{q^{1/2} - q^{-1/2}} = \frac{1-q^a}{1-q}q^{(1-a)/2}, \tag{1.2.54}$$

where $q = e^{2\pi i\sigma}$, it follows that

$$[a;\sigma]_n = \frac{(q^a;q)_n}{(1-q)^n}q^{n(1-a)/2-n(n-1)/4}, \tag{1.2.55}$$

and hence

$$\begin{aligned} &{}_rt_s(a_1, a_2, \ldots, a_r; b_1, \ldots, b_s; \sigma, z) \\ &= {}_r\phi_s(q^{a_1}, q^{a_2}, \ldots, q^{a_r}; q^{b_1}, \ldots, q^{b_s}; q, cz) \end{aligned} \tag{1.2.56}$$

with

$$c = (1-q)^{1+s-r}q^{r/2-s/2+(b_1+\cdots+b_s)/2-(a_1+\cdots+a_r)/2}, \tag{1.2.57}$$

which shows that the ${}_rt_s$ series is equivalent to the ${}_r\phi_s$ series in (1.2.49).

Elliptic numbers $[a;\sigma,\tau]$, which are a one-parameter generalization (deformation) of trigonometric numbers, are considered in §1.6, and the corresponding elliptic (and theta) hypergeometric series and their summation and transformation formulas are considered in Chapter 11.

We close this section with two identities involving ordinary binomial coefficients, which are particularly useful in handling some powers of q that arise in the derivations of many formulas containing q-series:

$$\binom{n+k}{2} = \binom{n}{2} + \binom{k}{2} + kn, \tag{1.2.58}$$

$$\binom{n-k}{2} = \binom{n}{2} + \binom{k}{2} + k - kn. \tag{1.2.59}$$

1.3 The q-binomial theorem

One of the most important summation formulas for hypergeometric series is given by the *binomial theorem*:

$${}_2F_1(a, c; c; z) = {}_1F_0(a; —; z) = \sum_{n=0}^{\infty}\frac{(a)_n}{n!}z^n = (1-z)^{-a}, \tag{1.3.1}$$

where $|z| < 1$. We shall show that this formula has the following q-analogue

$${}_1\phi_0(a; —; q, z) = \sum_{n=0}^{\infty}\frac{(a;q)_n}{(q;q)_n}z^n = \frac{(az;q)_\infty}{(z;q)_\infty}, \quad |z| < 1,\ |q| < 1, \tag{1.3.2}$$

which was derived by Cauchy [1843], Heine [1847] and by other mathematicians. See Askey [1980a], which also cites the books by Rothe [1811] and Schweins [1820], and the remark on p. 491 of Andrews, Askey, and Roy [1999] concerning the terminating form of the q-binomial theorem in Rothe [1811].

Heine's proof of (1.3.2), which can also be found in the books Heine [1878], Bailey [1935, p. 66] and Slater [1966, p. 92], is better understood if one first follows Askey's [1980a] approach of evaluating the sum of the binomial series in (1.3.1), and then carries out the analogous steps for the series in (1.3.2).

Let us set

$$f_a(z) = \sum_{n=0}^{\infty} \frac{(a)_n}{n!} z^n. \tag{1.3.3}$$

Since this series is uniformly convergent in $|z| \le \epsilon$ when $0 < \epsilon < 1$, we may differentiate it termwise to get

$$\begin{aligned} f_a'(z) &= \sum_{n=1}^{\infty} \frac{n(a)_n}{n!} z^{n-1} \\ &= \sum_{n=0}^{\infty} \frac{(a)_{n+1}}{n!} z^n = a f_{a+1}(z). \end{aligned} \tag{1.3.4}$$

Also

$$\begin{aligned} f_a(z) - f_{a+1}(z) &= \sum_{n=1}^{\infty} \frac{(a)_n - (a+1)_n}{n!} z^n \\ = \sum_{n=1}^{\infty} \frac{(a+1)_{n-1}}{n!} [a-(a+n)]\, z^n &= -\sum_{n=1}^{\infty} \frac{n(a+1)_{n-1}}{n!} z^n \\ = -\sum_{n=0}^{\infty} \frac{(a+1)_n}{n!} z^{n+1} &= -z f_{a+1}(z). \end{aligned} \tag{1.3.5}$$

Eliminating $f_{a+1}(z)$ from (1.3.4) and (1.3.5), we obtain the first order differential equation

$$f_a'(z) = \frac{a}{1-z} f_a(z), \tag{1.3.6}$$

subject to the initial condition $f_a(0) = 1$, which follows from the definition (1.3.3) of $f_a(z)$. Solving (1.3.6) under this condition immediately gives that $f_a(z) = (1-z)^{-a}$ for $|z| < 1$.

Analogously, let us now set

$$h_a(z) = \sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n} z^n, \quad |z| < 1,\ |q| < 1. \tag{1.3.7}$$

Clearly, $h_{q^a}(z) \to f_a(z)$ as $q \to 1$. Since $h_{aq}(z)$ is a q-analogue of $f_{a+1}(z)$, we first compute the difference

$$h_a(z) - h_{aq}(z) = \sum_{n=1}^{\infty} \frac{(a;q)_n - (aq;q)_n}{(q;q)_n} z^n$$

$$= \sum_{n=1}^{\infty} \frac{(aq;q)_{n-1}}{(q;q)_n} \left[1 - a - (1 - aq^n)\right] z^n$$

$$= -a \sum_{n=1}^{\infty} \frac{(1-q^n)(aq;q)_{n-1}}{(q;q)_n} z^n$$

$$= -a \sum_{n=1}^{\infty} \frac{(aq;q)_{n-1}}{(q;q)_{n-1}} z^n = -azh_{aq}(z), \tag{1.3.8}$$

giving an analogue of (1.3.5). Observing that

$$f'(z) = \lim_{q \to 1} \frac{f(z) - f(qz)}{(1-q)z} \tag{1.3.9}$$

for a differentiable function f, we next compute the difference

$$h_a(z) - h_a(qz) = \sum_{n=1}^{\infty} \frac{(a;q)_n}{(q;q)_n} (z^n - q^n z^n)$$

$$= \sum_{n=1}^{\infty} \frac{(a;q)_n}{(q;q)_{n-1}} z^n = \sum_{n=0}^{\infty} \frac{(a;q)_{n+1}}{(q;q)_n} z^{n+1}$$

$$= (1-a)zh_{aq}(z). \tag{1.3.10}$$

Eliminating $h_{aq}(z)$ from (1.3.8) and (1.3.10) gives

$$h_a(z) = \frac{1-az}{1-z} h_a(qz). \tag{1.3.11}$$

Iterating this relation $n-1$ times and then letting $n \to \infty$ we obtain

$$h_a(z) = \frac{(az;q)_n}{(z;q)_n} h_a(q^n z)$$

$$= \frac{(az;q)_\infty}{(z;q)_\infty} h_a(0) = \frac{(az;q)_\infty}{(z;q)_\infty}, \tag{1.3.12}$$

since $q^n \to 0$ as $n \to \infty$ and $h_a(0) = 1$ by (1.3.7), which completes the proof of (1.3.2).

One consequence of (1.3.2) is the product formula

$${}_1\phi_0(a;—;q,z)\ {}_1\phi_0(b;—;q,az) = {}_1\phi_0(ab;—;q,z), \tag{1.3.13}$$

which is a q-analogue of $(1-z)^{-a}(1-z)^{-b} = (1-z)^{-a-b}$.

In the special case $a = q^{-n}, n = 0, 1, 2, \ldots$, (1.3.2) gives

$${}_1\phi_0(q^{-n};—;q,z) = (zq^{-n};q)_n = (-z)^n q^{-n(n+1)/2}(q/z;q)_n, \tag{1.3.14}$$

where, by analytic continuation, z can be any complex number. From now on, unless stated othewise, whenever $q^{-j}, q^{-k}, q^{-m}, q^{-n}$ appear as numerator parameters in basic series it will be assumed that j, k, m, n, respectively, are nonnegative integers.

If we set $a = 0$ in (1.3.2), we get

$${}_1\phi_0(0;—;q,z) = \sum_{n=0}^{\infty} \frac{z^n}{(q;q)_n} = \frac{1}{(z;q)_\infty},\ |z| < 1, \tag{1.3.15}$$

which is a q-analogue of the exponential function e^z. Another q-analogue of e^z can be obtained from (1.3.2) by replacing z by $-z/a$ and then letting $a \to \infty$ to get

$$ {}_0\phi_0(—;—;q,-z) = \sum_{n=0}^{\infty} \frac{q^{n(n-1)/2}}{(q;q)_n} z^n = (-z;q)_\infty. \tag{1.3.16} $$

Observe that if we denote the q-*exponential* functions in (1.3.15) and (1.3.16) by $e_q(z)$ and $E_q(z)$, respectively, then $e_q(z)E_q(-z) = 1$, $e_{q^{-1}}(z) = E_q(-qz)$ by (1.2.24), and

$$ \lim_{q\to 1^-} e_q(z(1-q)) = \lim_{q\to 1^-} E_q(z(1-q)) = e^z. \tag{1.3.17} $$

In deriving q-analogues of various formulas we shall sometimes use the observation that

$$ \frac{(q^a z;q)_\infty}{(z;q)_\infty} = {}_1\phi_0(q^a;—;q,z) \to {}_1F_0(a;—;z) = (1-z)^{-a} \text{ as } q \to 1^-. \tag{1.3.18} $$

Thus

$$ \lim_{q\to 1^-} \frac{(q^a z;q)_\infty}{(z;q)_\infty} = (1-z)^{-a}, \quad |z| < 1, \quad a \text{ real}. \tag{1.3.19} $$

By analytic continuation this holds for z in the complex plane cut along the positive real axis from 1 to ∞, with $(1-z)^{-a}$ positive when z is real and less than 1.

Let Δ and ∇ be the *forward* and *backward* q-*difference operators*, respectively, defined by

$$ \Delta f(z) = f(qz) - f(z), \quad \nabla f(z) = f(q^{-1}z) - f(z), \tag{1.3.20} $$

where we take $0 < q < 1$, without any loss of generality. Then the unique analytic solutions of

$$ \frac{\Delta f(z)}{\Delta z} = f(z), \quad f(0) = 1 \quad \text{and} \quad \frac{\nabla g(z)}{\nabla z} = g(z), \quad g(0) = 1, \tag{1.3.21} $$

are

$$ f(z) = e_q(z(1-q)) \quad \text{and} \quad g(z) = E_q(z(1-q)). \tag{1.3.22} $$

The *symmetric* q-*difference operator* δ_q is defined by

$$ \delta_q f(z) = f(zq^{1/2}) - f(zq^{-1/2}). \tag{1.3.23} $$

If we seek an analytic solution of the initial-value problem

$$ \frac{\delta_q f(z)}{\delta_q z} = f(z), \quad f(0) = 1, \tag{1.3.24} $$

in the form $\sum_{n=0}^{\infty} a_n z^n$, then we find that

$$ a_{n+1} = \frac{1-q}{1-q^{n+1}} q^{n/2} a_n, \quad a_0 = 1, \tag{1.3.25} $$

$n = 0, 1, 2, \ldots$. Hence, $a_n = (1-q)^n q^{(n^2-n)/4}/(q;q)_n$, and we have a third q-exponential function

$$\exp_q(z) = \sum_{n=0}^{\infty} \frac{(1-q)^n q^{(n^2-n)/4}}{(q;q)_n} z^n = \sum_{n=0}^{\infty} \frac{1}{[n;\sigma]!} z^n \tag{1.3.26}$$

with $q = e^{2\pi i\sigma}$. This q-exponential function has the properties

$$\exp_{q^{-1}}(z) = \exp_q(z), \qquad \lim_{q\to 1} \exp_q(z) = e^z, \tag{1.3.27}$$

and it is an entire function of z of order zero with an infinite product representation in terms of its zeros. See Nelson and Gartley [1994], and Atakishiyev and Suslov [1992a]. The multi-sheet Riemann surface associated with the *q-logarithm* inverse function $z = \ln_q(w)$ of $w = \exp_q(z)$ is considered in Nelson and Gartley [1996].

Ismail and Zhang [1994] found an extension of $\exp_q(z)$ in the form

$$f(z) = \sum_{m=0}^{\infty} \frac{q^{m^2/4}}{(q;q)_m} \left(aq^{\frac{1-m}{2}+z}, aq^{\frac{1-m}{2}-z}; q\right)_m b^m, \tag{1.3.28}$$

which has the property

$$\frac{\delta f(z)}{\delta x(z)} = f(z), \quad \delta f(z) = f(z+1/2) - f(z-1/2), \tag{1.3.29}$$

where

$$x(z) = C(q^z + q^{-z}) \tag{1.3.30}$$

with $C = -abq^{1/4}/(1-q)$ is the so-called *q-quadratic lattice*, and a and b are arbitrary complex parameters such that $|ab| < 1$. In the particular case $q^z = e^{-i\theta}$, $0 \le \theta \le \pi$, $x = \cos\theta$, the q-exponential function in (1.3.28) becomes the function

$$\mathcal{E}_q(x;a,b) = \sum_{m=0}^{\infty} \frac{q^{m^2/4}}{(q;q)_m} \left(q^{\frac{1-m}{2}} ae^{i\theta}, q^{\frac{1-m}{2}} ae^{-i\theta}; q\right)_m b^m. \tag{1.3.31}$$

Ismail and Zhang showed that

$$\lim_{q\to 1} \mathcal{E}_q(x;a,b(1-q)) = \exp[(1+a^2-2ax)b], \tag{1.3.32}$$

and that $\mathcal{E}_q(x;a,b)$ is an entire function of x when $|ab| < 1$. From (1.3.32) they observed that $\mathcal{E}_q(x;-i,-it/2)$ is a q-analogue of e^{xt}. It is now standard to use the notation in Suslov [2003] for the slightly modified q-exponential function

$$\mathcal{E}_q(x;\alpha) = \frac{(\alpha^2;q^2)_\infty}{(q\alpha^2;q^2)_\infty} \sum_{m=0}^{\infty} \frac{q^{m^2/4}}{(q;q)_m} (-i\alpha)^m \left(-iq^{\frac{1-m}{2}} e^{i\theta}, -iq^{\frac{1-m}{2}} e^{-i\theta}; q\right)_m, \tag{1.3.33}$$

which, because of the normalizing factor that he introduced, has the nice property that $\mathcal{E}_q(0;\alpha) = 1$ (see Suslov [2003, p. 17]).

1.4 Heine's transformation formulas for ${}_2\phi_1$ series

Heine [1847, 1878] showed that

$$ {}_2\phi_1(a,b;c;q,z) = \frac{(b,az;q)_\infty}{(c,z;q)_\infty}\,{}_2\phi_1(c/b,z;az;q,b), \tag{1.4.1}$$

where $|z|<1$ and $|b|<1$. To prove this transformation formula, first observe from the q-binomial theorem (1.3.2) that

$$\frac{(cq^n;q)_\infty}{(bq^n;q)_\infty} = \sum_{m=0}^{\infty} \frac{(c/b;q)_m}{(q;q)_m}(bq^n)^m.$$

Hence, for $|z|<1$ and $|b|<1$,

$$\begin{aligned}
{}_2\phi_1(a,b;c;q,z) &= \frac{(b;q)_\infty}{(c;q)_\infty}\sum_{n=0}^{\infty}\frac{(a;q)_n(cq^n;q)_\infty}{(q;q)_n(bq^n;q)_\infty}z^n \\
&= \frac{(b;q)_\infty}{(c;q)_\infty}\sum_{n=0}^{\infty}\frac{(a;q)_n}{(q;q)_n}z^n\sum_{m=0}^{\infty}\frac{(c/b;q)_m}{(q;q)_m}(bq^n)^m \\
&= \frac{(b;q)_\infty}{(c;q)_\infty}\sum_{m=0}^{\infty}\frac{(c/b;q)_m}{(q;q)_m}b^m\sum_{n=0}^{\infty}\frac{(a;q)_n}{(q;q)_n}(zq^m)^n \\
&= \frac{(b;q)_\infty}{(c;q)_\infty}\sum_{m=0}^{\infty}\frac{(c/b;q)_m}{(q;q)_m}b^m\frac{(azq^m;q)_\infty}{(zq^m;q)_\infty} \\
&= \frac{(b,az;q)_\infty}{(c,z;q)_\infty}\,{}_2\phi_1(c/b,z;az;q,b)
\end{aligned}$$

by (1.3.2), which gives (1.4.1).

Heine also showed that Euler's transformation formula

$$ {}_2F_1(a,b;c;z) = (1-z)^{c-a-b}\,{}_2F_1(c-a,c-b;c;z) \tag{1.4.2}$$

has a q-analogue of the form

$$ {}_2\phi_1(a,b;c;q,z) = \frac{(abz/c;q)_\infty}{(z;q)_\infty}\,{}_2\phi_1(c/a,c/b;c;q,abz/c). \tag{1.4.3}$$

A short way to prove this formula is just to iterate (1.4.1) as follows

$$ {}_2\phi_1(a,b;c;q,z) = \frac{(b,az;q)_\infty}{(c,z;q)_\infty}\,{}_2\phi_1(c/b,z;az;q,b) \tag{1.4.4}$$

$$ = \frac{(c/b,bz;q)_\infty}{(c,z;q)_\infty}\,{}_2\phi_1(abz/c,b;bz;q,c/b) \tag{1.4.5}$$

$$ = \frac{(abz/c;q)_\infty}{(z;q)_\infty}\,{}_2\phi_1(c/a,c/b;c;q,abz/c). \tag{1.4.6}$$

1.5 Heine's q-analogue of Gauss' summation formula

In order to derive Heine's [1847] q-analogue of Gauss' summation formula (1.2.11) it suffices to set $z = c/ab$ in (1.4.1), assume that $|b| < 1$, $|c/ab| < 1$, and observe that the series on the right side of

$$ {}_2\phi_1(a, b; c; q, c/ab) = \frac{(b, c/b; q)_\infty}{(c, c/ab; q)_\infty}\, {}_1\phi_0(c/ab; —; q, b) $$

can be summed by (1.3.2) to give

$$ {}_2\phi_1(a, b; c; q, c/ab) = \frac{(c/a, c/b; q)_\infty}{(c, c/ab; q)_\infty}. \tag{1.5.1} $$

By analytic continuation, we may drop the assumption that $|b| < 1$ and require only that $|c/ab| < 1$ for (1.5.1) to be valid.

For the terminating case when $a = q^{-n}$, (1.5.1) reduces to

$$ {}_2\phi_1(q^{-n}, b; c; q, cq^n/b) = \frac{(c/b; q)_n}{(c; q)_n}. \tag{1.5.2} $$

By inversion or by changing the order of summation it follows from (1.5.2) that

$$ {}_2\phi_1(q^{-n}, b; c; q, q) = \frac{(c/b; q)_n}{(c; q)_n} b^n. \tag{1.5.3} $$

Both (1.5.2) and (1.5.3) are q-analogues of Vandermonde's formula (1.2.9). These formulas can be used to derive other important formulas such as, for example, Jackson's [1910a] transformation formula

$$ \begin{aligned} {}_2\phi_1(a, b; c; q, z) &= \frac{(az; q)_\infty}{(z; q)_\infty} \sum_{k=0}^{\infty} \frac{(a, c/b; q)_k}{(q, c, az; q)_k} (-bz)^k q^{\binom{k}{2}} \\ &= \frac{(az; q)_\infty}{(z; q)_\infty}\, {}_2\phi_2(a, c/b; c, az; q, bz). \end{aligned} \tag{1.5.4} $$

This formula is a q-analogue of the Pfaff–Kummer transformation formula

$$ {}_2F_1(a, b; c; z) = (1 - z)^{-a}\, {}_2F_1(a, c - b; c; z/(z - 1)). \tag{1.5.5} $$

To prove (1.5.4), we use (1.5.2) to write

$$ \frac{(b; q)_k}{(c; q)_k} = \sum_{n=0}^{k} \frac{(q^{-k}, c/b; q)_n}{(q, c; q)_n} \left(bq^k\right)^n $$

and hence

$$\begin{aligned}
&{}_2\phi_1(a,b;c;q,z)\\
&=\sum_{k=0}^{\infty}\frac{(a;q)_k}{(q;q)_k}z^k\sum_{n=0}^{k}\frac{(q^{-k},c/b;q)_n}{(q,c;q)_n}\left(bq^k\right)^n\\
&=\sum_{n=0}^{\infty}\sum_{k=n}^{\infty}\frac{(a;q)_k(c/b;q)_n}{(q;q)_{k-n}(q,c;q)_n}z^k(-b)^nq^{\binom{n}{2}}\\
&=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(a;q)_{k+n}(c/b;q)_n}{(q;q)_k(q,c;q)_n}(-bz)^nz^kq^{\binom{n}{2}}\\
&=\sum_{n=0}^{\infty}\frac{(a,c/b;q)_n}{(q,c;q)_n}(-bz)^nq^{\binom{n}{2}}\sum_{k=0}^{\infty}\frac{(aq^n;q)_k}{(q;q)_k}z^k\\
&=\frac{(az;q)_\infty}{(z;q)_\infty}\sum_{n=0}^{\infty}\frac{(a,c/b;q)_n}{(q,c,az;q)_n}(-bz)^nq^{\binom{n}{2}},
\end{aligned}$$

by (1.3.2). Also see Andrews [1973]. If $a=q^{-n}$, then the series on the right side of (1.5.4) can be reversed (by replacing k by $n-k$) to yield Sears' [1951c] transformation formula

$$\begin{aligned}
&{}_2\phi_1(q^{-n},b;c;q,z)\\
&=\frac{(c/b;q)_n}{(c;q)_n}\left(\frac{bz}{q}\right)^n{}_3\phi_2(q^{-n},q/z,c^{-1}q^{1-n};bc^{-1}q^{1-n},0;q,q).
\end{aligned}\tag{1.5.6}$$

1.6 Jacobi's triple product identity, theta functions, and elliptic numbers

Jacobi's [1829] well-known *triple product identity* (see Andrews [1971])

$$(zq^{\frac12},q^{\frac12}/z,q;q)_\infty=\sum_{n=-\infty}^{\infty}(-1)^nq^{n^2/2}z^n,\quad z\neq 0,\tag{1.6.1}$$

can be easily derived by using Heine's summation formula (1.5.1).

First, set $c=bzq^{\frac12}$ in (1.5.1) and then let $b\to 0$ and $a\to\infty$ to obtain

$$\sum_{n=0}^{\infty}\frac{(-1)^nq^{n^2/2}}{(q;q)_n}z^n=(zq^{\frac12};q)_\infty.\tag{1.6.2}$$

Similarly, setting $c=zq$ in (1.5.1) and letting $a\to\infty$ and $b\to\infty$ we get

$$\sum_{n=0}^{\infty}\frac{q^{n^2}z^n}{(q,zq;q)_n}=\frac{1}{(zq;q)_\infty}.\tag{1.6.3}$$

Now use (1.6.2) to find that

$$\begin{aligned}
&(zq^{\frac12},q^{\frac12}/z;q)_\infty\\
&=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^{m+n}q^{(m^2+n^2)/2}}{(q;q)_m(q;q)_n}z^{m-n}
\end{aligned}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2/2}}{(q;q)_n} z^n \sum_{k=0}^{\infty} \frac{q^{k^2}}{(q,q^{n+1};q)_k} q^{nk}$$
$$+ \sum_{n=1}^{\infty} \frac{(-1)^n q^{n^2/2}}{(q;q)_n} z^{-n} \sum_{k=0}^{\infty} \frac{q^{k^2}}{(q,q^{n+1};q)_k} q^{nk}. \tag{1.6.4}$$

Formula (1.6.1) then follows from (1.6.3) by observing that

$$\frac{1}{(q;q)_n} \sum_{k=0}^{\infty} \frac{q^{k^2}}{(q,q^{n+1};q)_k} q^{nk} = \frac{1}{(q;q)_n (q^{n+1};q)_\infty} = \frac{1}{(q;q)_\infty}.$$

An important application of (1.6.1) is that it can be used to express the *theta functions* (Whittaker and Watson [1965, Chapter 21])

$$\vartheta_1(x,q) = 2\sum_{n=0}^{\infty} (-1)^n q^{(n+1/2)^2} \sin(2n+1)x, \tag{1.6.5}$$

$$\vartheta_2(x,q) = 2\sum_{n=0}^{\infty} q^{(n+1/2)^2} \cos(2n+1)x, \tag{1.6.6}$$

$$\vartheta_3(x,q) = 1 + 2\sum_{n=1}^{\infty} q^{n^2} \cos 2nx, \tag{1.6.7}$$

$$\vartheta_4(x,q) = 1 + 2\sum_{n=1}^{\infty} (-1)^n q^{n^2} \cos 2nx \tag{1.6.8}$$

in terms of infinite products. Just replace q by q^2 in (1.6.1) and then set z equal to $qe^{2ix}, -qe^{2ix}, -e^{2ix}, e^{2ix}$, respectively, to obtain

$$\vartheta_1(x,q) = 2q^{1/4} \sin x \prod_{n=1}^{\infty} (1-q^{2n})(1-2q^{2n}\cos 2x + q^{4n}), \tag{1.6.9}$$

$$\vartheta_2(x,q) = 2q^{1/4} \cos x \prod_{n=1}^{\infty} (1-q^{2n})(1+2q^{2n}\cos 2x + q^{4n}), \tag{1.6.10}$$

$$\vartheta_3(x,q) = \prod_{n=1}^{\infty} (1-q^{2n})(1+2q^{2n-1}\cos 2x + q^{4n-2}), \tag{1.6.11}$$

and

$$\vartheta_4(x,q) = \prod_{n=1}^{\infty} (1-q^{2n})(1-2q^{2n-1}\cos 2x + q^{4n-2}). \tag{1.6.12}$$

It is common to write $\vartheta_k(x)$ for $\vartheta_k(x,q)$, $k = 1, \ldots, 4$.

Since, from (1.6.9) and (1.6.10),

$$\lim_{q\to 0} 2^{-1} q^{-\frac{1}{4}} \vartheta_1(x,q) = \sin x, \quad \lim_{q\to 0} 2^{-1} q^{-\frac{1}{4}} \vartheta_2(x,q) = \cos x, \tag{1.6.13}$$

one can think of the theta functions $\vartheta_1(x,q)$ and $\vartheta_2(x,q)$ as one-parameter deformations (generalizations) of the trigonometric functions $\sin x$ and $\cos x$, respectively. This led Frenkel and Turaev [1995] to define an *elliptic number* $[a;\sigma,\tau]$ by

$$[a;\sigma,\tau]=\frac{\vartheta_1(\pi\sigma a,e^{\pi i\tau})}{\vartheta_1(\pi\sigma,e^{\pi i\tau})},\tag{1.6.14}$$

where a is a complex number and the *modular parameters* σ and τ are fixed complex numbers such that Im $(\tau)>0$ and $\sigma\neq m+n\tau$ for integer values of m and n, so that the denominator $\vartheta_1(\pi\sigma,e^{\pi i\tau})$ in (1.6.14) is never zero. Then, from (1.6.9) it is clear that $[a;\sigma,\tau]$ is well-defined, $[-a;\sigma,\tau]=-[a;\sigma,\tau]$, $[1;\sigma,\tau]=1$, and

$$\lim_{\mathrm{I}\mu\ \tau\to\infty}[a;\sigma,\tau]=\frac{\sin(\pi\sigma a)}{\sin(\pi\sigma)}=[a;\sigma].\tag{1.6.15}$$

Hence, the elliptic number $[a;\sigma,\tau]$ is a one-parameter deformation of the trigonometric number $[a;\sigma]$ and a two-parameter deformation of the number a. Notice that $[a;\sigma,\tau]$ is called an "elliptic number" even though it is not an elliptic (doubly periodic and meromorphic) function of a. However, $[a;\sigma,\tau]$ is a quotient of ϑ_1 functions and, as is well-known (see Whittaker and Watson [1965, §21.5]), any (doubly periodic meromorphic) elliptic function can be written as a constant multiple of a quotient of products of ϑ_1 functions. The corresponding elliptic hypergeometric series are considered in Chapter 11.

1.7 A q-analogue of Saalschütz's summation formula

Pfaff [1797] discovered the summation formula

$${}_3F_2(a,b,-n;c,1+a+b-c-n;1)=\frac{(c-a)_n(c-b)_n}{(c)_n(c-a-b)_n},\quad n=0,1,\ldots,\tag{1.7.1}$$

which sums a terminating balanced ${}_3F_2(1)$ series with argument 1. It was rediscovered by Saalschütz [1890] and is usually called *Saalschütz formula* or the *Pfaff–Saalschütz formula*; see Askey [1975]. To derive a q-analogue of (1.7.1), observe that since, by (1.3.2),

$$\frac{(abz/c;q)_\infty}{(z;q)_\infty}=\sum_{k=0}^{\infty}\frac{(ab/c;q)_k}{(q;q)_k}z^k$$

the right side of (1.4.3) equals

$$\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\frac{(ab/c;q)_k(c/a,c/b;q)_m}{(q;q)_k(q,c;q)_m}\left(\frac{ab}{c}\right)^m z^{k+m},$$

and hence, equating the coefficients of z^n on both sides of (1.4.3) we get

$$\sum_{j=0}^{n}\frac{(q^{-n},c/a,c/b;q)_j}{(q,c,cq^{1-n}/ab;q)_j}q^j=\frac{(a,b;q)_n}{(c,ab/c;q)_n}.$$

Replacing a,b by $c/a,c/b$, respectively, this gives the following sum of a terminating balanced ${}_3\phi_2$ series

$${}_3\phi_2(a,b,q^{-n};c,abc^{-1}q^{1-n};q,q)=\frac{(c/a,c/b;q)_n}{(c,c/ab;q)_n},\quad n=0,1,\ldots,\tag{1.7.2}$$

which was first derived by Jackson [1910a]. It is easy to see that (1.7.1) follows from (1.7.2) by replacing a, b, c in (1.7.2) by q^a, q^b, q^c, respectively, and letting $q \to 1$. Note that letting $a \to \infty$ in (1.7.2) gives (1.5.2), while letting $a \to 0$ gives (1.5.3).

1.8 The Bailey–Daum summation formula

Bailey [1941] and Daum [1942] independently discovered the summation formula

$$ {}_2\phi_1(a, b; aq/b; q, -q/b) = \frac{(-q;q)_\infty (aq, aq^2/b^2; q^2)_\infty}{(aq/b, -q/b; q)_\infty}, \tag{1.8.1} $$

which is a q-analogue of Kummer's formula

$$ {}_2F_1(a, b; 1+a-b; -1) = \frac{\Gamma(1+a-b)\Gamma(1+\frac{1}{2}a)}{\Gamma(1+a)\Gamma(1+\frac{1}{2}a-b)}. \tag{1.8.2} $$

Formula (1.8.1) can be easily obtained from (1.4.1) by using the identity (1.2.40) and a limiting form of (1.2.39), namely, $(a;q)_\infty = (a, aq; q^2)_\infty$, to see that

$$ \begin{aligned} &{}_2\phi_1(a, b; aq/b; q, -q/b) \\ &= \frac{(a, -q; q)_\infty}{(aq/b, -q/b; q)_\infty} \, {}_2\phi_1(q/b, -q/b; -q; q, a) \\ &= \frac{(a, -q; q)_\infty}{(aq/b, -q/b; q)_\infty} \sum_{n=0}^\infty \frac{(q^2/b^2; q^2)_n}{(q^2; q^2)_n} a^n \\ &= \frac{(a, -q; q)_\infty}{(aq/b, -q/b; q)_\infty} \frac{(aq^2/b^2; q^2)_\infty}{(a; q^2)_\infty} \qquad \text{by (1.3.2)} \\ &= \frac{(-q;q)_\infty (aq, aq^2/b^2; q^2)_\infty}{(aq/b, -q/b; q)_\infty}. \end{aligned} $$

1.9 q-analogues of the Karlsson–Minton summation formulas

Minton [1970] showed that if a is a negative integer and $m_1, m_2, \ldots, m_r$ are nonnegative integers such that $-a \geq m_1 + \cdots + m_r$, then

$$ \begin{aligned} &{}_{r+2}F_{r+1}\left[\begin{matrix} a, b, b_1+m_1, \ldots, b_r+m_r \\ b+1, b_1, \ldots, b_r \end{matrix}; 1\right] \\ &= \frac{\Gamma(b+1)\Gamma(1-a)}{\Gamma(1+b-a)} \frac{(b_1-b)_{m_1} \cdots (b_r-b)_{m_r}}{(b_1)_{m_1} \cdots (b_r)_{m_r}} \end{aligned} \tag{1.9.1} $$

where, as usual, it is assumed that none of the factors in the denominators of the terms of the series is zero. Karlsson [1971] showed that (1.9.1) also holds when a is not a negative integer provided that the series converges, i.e., if $\mathrm{Re}(-a) > m_1 + \cdots + m_r - 1$, and he deduced from (1.9.1) that

$$ {}_{r+1}F_r\left[\begin{matrix} a, b_1+m_1, \ldots, b_r+m_r \\ b_1, \ldots, b_r \end{matrix}; 1\right] = 0, \quad \mathrm{Re}\,(-a) > m_1 + \cdots + m_r, \tag{1.9.2} $$

$$
\begin{aligned}
&{}_{r+1}F_r\left[\begin{matrix}-(m_1+\cdots+m_r), b_1+m_1, \ldots, b_r+m_r\\ b_1, \ldots, b_r\end{matrix}; 1\right]\\
&= (-1)^{m_1+\cdots+m_r}\frac{(m_1+\cdots+m_r)!}{(b_1)_{m_1}\cdots(b_r)_{m_r}}.
\end{aligned} \tag{1.9.3}
$$

These formulas are particularly useful for evaluating sums that appear as solutions to some problems in theoretical physics such as the Racah coefficients. They were also used by Gasper [1981b] to prove the orthogonality on $(0, 2\pi)$ of certain functions that arose in Greiner's [1980] work on spherical harmonics on the Heisenberg group. Here we shall present Gasper's [1981a] derivation of q-analogues of the above formulas. Some of the formulas derived below will be used in Chapter 7 to prove the orthogonality relation for the continuous q-ultraspherical polynomials.

Observe that if m and n are nonnegative integers with $m \geq n$, then

$$
{}_2\phi_1(q^{-n}, q^{-m}; b_r; q, q) = \frac{(b_r q^m; q)_n}{(b_r; q)_n} q^{-mn}
$$

by (1.5.3), and hence

$$
\begin{aligned}
&{}_{r+1}\phi_r\left[\begin{matrix}a_1, \ldots, a_r, b_r q^m\\ b_1, \ldots, b_{r-1}, b_r\end{matrix}; q, z\right]\\
&= \sum_{n=0}^{\infty}\frac{(a_1, \ldots, a_r; q)_n}{(q, b_1, \ldots, b_{r-1}; q)_n} z^n \sum_{k=0}^{n}\frac{(q^{-n}, q^{-m}; q)_k}{(q, b_r; q)_k} q^{mn+k}\\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{m}\frac{(a_1, \ldots, a_r; q)_n (q^{-m}; q)_k}{(b_1, \ldots, b_{r-1}; q)_n (q; q)_{n-k}(q, b_r; q)_k} z^n (-1)^k q^{mn+k-nk+\binom{k}{2}}\\
&= \sum_{k=0}^{m}\frac{(q^{-m}, a_1, \ldots, a_r; q)_k}{(q, b_1, \ldots, b_r; q)_k}(-zq^m)^k q^{-\binom{k}{2}}\\
&\quad\times {}_r\phi_{r-1}\left[\begin{matrix}a_1 q^k, \ldots, a_r q^k\\ b_1 q^k, \ldots, b_{r-1} q^k\end{matrix}; q, zq^{m-k}\right], \qquad |z| < 1.
\end{aligned} \tag{1.9.4}
$$

This expansion formula is a q-analogue of a formula in Fox [1927, (1.11)] and independently derived by Minton [1970, (4)].

When $r = 2$, formulas (1.9.4), (1.5.1) and (1.5.3) yield

$$
\begin{aligned}
&{}_3\phi_2\left[\begin{matrix}a, b, b_1 q^m\\ bq, b_1\end{matrix}; q, a^{-1}q^{1-m}\right] = \frac{(q, bq/a; q)_\infty}{(bq, q/a; q)_\infty}\,{}_2\phi_1(q^{-m}, b; b_1; q, q)\\
&= \frac{(q, bq/a; q)_\infty (b_1/b; q)_m}{(bq, q/a; q)_\infty (b_1; q)_m} b^m,
\end{aligned} \tag{1.9.5}
$$

provided that $|a^{-1}q^{1-m}| < 1$. By induction it follows from (1.9.4) and (1.9.5) that if $m_1, \ldots, m_r$ are nonnegative integers and $|a^{-1}q^{1-(m_1+\cdots+m_r)}| < 1$, then

$$
\begin{aligned}
&{}_{r+2}\phi_{r+1}\left[\begin{matrix}a, b, b_1 q^{m_1}, \ldots, b_r q^{m_r}\\ bq, b_1, \ldots, b_r\end{matrix}; q, a^{-1}q^{1-(m_1+\cdots+m_r)}\right]\\
&= \frac{(q, bq/a; q)_\infty}{(bq, q/a; q)_\infty}\frac{(b_1/b; q)_{m_1}\cdots(b_r/b; q)_{m_r}}{(b_1; q)_{m_1}\cdots(b_r; q)_{m_r}} b^{m_1+\cdots+m_r}
\end{aligned} \tag{1.9.6}
$$

which is a q-analogue of (1.9.1). Formula (1.9.1) can be derived from (1.9.6) by replacing $a, b, b_1, \ldots, b_r$ by $q^a, q^b, q^{b_1}, \ldots, q^{b_r}$, respectively, and letting $q \to 1$.

Setting $b_r = b, m_r = 1$ and then replacing r by $r+1$ in (1.9.6) gives

$$
{}_{r+1}\phi_r \left[\begin{matrix} a, b_1 q^{m_1}, \ldots, b_r q^{m_r} \\ b_1, \ldots, b_r \end{matrix} ; q, a^{-1} q^{-(m_1+ \cdots + m_r)} \right] = 0, \ |a^{-1} q^{-(m_1+ \cdots + m_r)}| < 1, \tag{1.9.7}
$$

while letting $b \to \infty$ in the case $a = q^{-(m_1+ \cdots + m_r)}$ of (1.9.6) gives

$$
\begin{aligned}
&{}_{r+1}\phi_r \left[\begin{matrix} q^{-(m_1+ \cdots + m_r)}, b_1 q^{m_1}, \ldots, b_r q^{m_r} \\ b_1, \ldots, b_r \end{matrix} ; q, 1 \right] \\
&= \frac{(-1)^{m_1+ \cdots + m_r} (q;q)_{m_1+ \cdots + m_r}}{(b_1;q)_{m_1} \cdots (b_r;q)_{m_r}} q^{-(m_1+ \cdots + m_r)(m_1+ \cdots + m_r+1)/2},
\end{aligned} \tag{1.9.8}
$$

which are q-analogues of (1.9.2) and (1.9.3). Another q-analogue of (1.9.3) can be derived by letting $b \to 0$ in (1.9.6) to obtain

$$
\begin{aligned}
&{}_{r+1}\phi_r \left[\begin{matrix} a, b_1 q^{m_1}, \ldots, b_r q^{m_r} \\ b_1, \ldots, b_r \end{matrix} ; q, a^{-1} q^{1-(m_1+ \cdots + m_r)} \right] \\
&= \frac{(-1)^{m_1+ \cdots + m_r} (q;q)_\infty b_1^{m_1} \cdots b_r^{m_r}}{(q/a;q)_\infty (b_1;q)_{m_1} \cdots (b_r;q)_{m_r}} q^{\binom{m_1}{2} + \cdots + \binom{m_r}{2}},
\end{aligned} \tag{1.9.9}
$$

when $|a^{-1} q^{1-(m_1+ \cdots + m_r)}| < 1$.

In addition, if $a = q^{-n}$ and n is a nonnegative integer then we can reverse the order of summation of the series in (1.9.6), (1.9.7) and (1.9.9) to obtain

$$
\begin{aligned}
&{}_{r+2}\phi_{r+1} \left[\begin{matrix} q^{-n}, b, b_1 q^{m_1}, \ldots, b_r q^{m_r} \\ bq, b_1, \ldots, b_r \end{matrix} ; q, q \right] \\
&= \frac{b^n (q;q)_n (b_1/b;q)_{m_1} \cdots (b_r/b;q)_{m_r}}{(bq;q)_n (b_1;q)_{m_1} \cdots (b_r;q)_{m_r}}, \quad n \geq m_1 + \cdots + m_r,
\end{aligned} \tag{1.9.10}
$$

$$
{}_{r+1}\phi_r \left[\begin{matrix} q^{-n}, b_1 q^{m_1}, \ldots, b_r q^{m_r} \\ b_1, \ldots, b_r \end{matrix} ; q, q \right] = 0, \quad n > m_1 + \cdots + m_r, \tag{1.9.11}
$$

and the following generalization of (1.9.8)

$$
{}_{r+1}\phi_r \left[\begin{matrix} q^{-n}, b_1 q^{m_1}, \ldots, b_r q^{m_r} \\ b_1, \ldots, b_r \end{matrix} ; q, 1 \right] = \frac{(-1)^n (q;q)_n q^{-n(n+1)/2}}{(b_1;q)_{m_1} \cdots (b_r;q)_{m_r}}, \tag{1.9.12}
$$

where $n \geq m_1 + \cdots + m_r$, which also follows by letting $b \to \infty$ in (1.9.10). Note that the $b \to 0$ limit case of (1.9.10) is (1.9.11) when $n > m_1 + \cdots + m_r$, and it is the $a = q^{-(m_1+ \cdots + m_r)}$ special case of (1.9.9) when $n = m_1 + \cdots + m_r$.

1.10 The q-gamma and q-beta functions

The *q-gamma* function

$$
\Gamma_q(x) = \frac{(q;q)_\infty}{(q^x;q)_\infty} (1-q)^{1-x}, \ 0 < q < 1, \tag{1.10.1}
$$

was introduced by Thomae [1869] and later by Jackson [1904e]. Heine [1847] gave an equivalent definition, but without the factor $(1-q)^{1-x}$. When $x = n+1$ with n a nonnegative integer, this definition reduces to

$$\Gamma_q(n+1) = 1(1+q)(1+q+q^2)\cdots(1+q+\cdots+q^{n-1}), \tag{1.10.2}$$

which clearly approaches $n!$ as $q \to 1^-$. Hence $\Gamma_q(n+1)$ tends to $\Gamma(n+1) = n!$ as $q \to 1^-$. The definition of $\Gamma_q(x)$ can be extended to $|q| < 1$ by using the principal values of q^x and $(1-q)^{1-x}$ in (1.10.1).

To show that

$$\lim_{q\to 1^-} \Gamma_q(x) = \Gamma(x) \tag{1.10.3}$$

we shall give a simple, formal proof due to Gosper; see Andrews [1986]. From (1.10.1),

$$\begin{aligned}\Gamma_q(x+1) &= \frac{(q;q)_\infty}{(q^{x+1};q)_\infty}(1-q)^{-x} \\ &= \prod_{n=1}^{\infty} \frac{(1-q^n)(1-q^{n+1})^x}{(1-q^{n+x})(1-q^n)^x}.\end{aligned}$$

Hence

$$\begin{aligned}\lim_{q\to 1^-} \Gamma_q(x+1) &= \prod_{n=1}^{\infty} \frac{n}{n+x}\left(\frac{n+1}{n}\right)^x \\ &= x\left[x^{-1}\prod_{n=1}^{\infty}\left(1+\frac{x}{n}\right)^{-1}\left(1+\frac{1}{n}\right)^x\right] \\ &= x\Gamma(x) = \Gamma(x+1)\end{aligned}$$

by Euler's product formula (see Whittaker and Watson [1965, §12.11]) and the well-known functional equation for the gamma function

$$\Gamma(x+1) = x\,\Gamma(x), \quad \Gamma(1) = 1. \tag{1.10.4}$$

For a rigorous justification of the above steps see Koornwinder [1990]. From (1.10.1) it is easily seen that, analogous to (1.10.4), $\Gamma_q(x)$ satisfies the functional equation

$$f(x+1) = \frac{1-q^x}{1-q} f(x), \quad f(1) = 1. \tag{1.10.5}$$

Askey [1978] derived analogues of many of the well-known properties of the gamma function, including its log-convexity (see the exercises at the end of this chapter), which show that (1.10.1) is a natural q-analogue of $\Gamma(x)$.

It is obvious from (1.10.1) that $\Gamma_q(x)$ has poles at $x = 0, -1, -2, \ldots$. The residue at $x = -n$ is

$$\begin{aligned}\lim_{x\to -n}(x+n)\Gamma_q(x) &= \frac{(1-q)^{n+1}}{(1-q^{-n})(1-q^{1-n})\cdots(1-q^{-1})} \lim_{x\to -n} \frac{x+n}{1-q^{x+n}} \\ &= \frac{(1-q)^{n+1}}{(q^{-n};q)_n \log q^{-1}}.\end{aligned} \tag{1.10.6}$$

The q-gamma function has no zeros, so its reciprocal is an entire function with zeros at $x = 0, -1, -2, \ldots$. Since

$$\frac{1}{\Gamma_q(x)} = (1-q)^{x-1} \prod_{n=0}^{\infty} \frac{1-q^{n+x}}{1-q^{n+1}}, \tag{1.10.7}$$

the function $1/\Gamma_q(x)$ has zeros at $x = -n \pm 2\pi i k/\log q$, where k and n are nonnegative integers.

A q-analogue of Legendre's duplication formula

$$\Gamma(2x)\Gamma\left(\frac{1}{2}\right) = 2^{2x-1}\Gamma(x)\Gamma\left(x+\frac{1}{2}\right) \tag{1.10.8}$$

can be easily derived by observing that

$$\begin{aligned} \frac{\Gamma_{q^2}(x)\Gamma_{q^2}\left(x+\frac{1}{2}\right)}{\Gamma_{q^2}\left(\frac{1}{2}\right)} &= \frac{(q, q^2; q^2)_\infty}{(q^{2x}, q^{2x+1}; q^2)_\infty} (1-q^2)^{1-2x} \\ &= \frac{(q;q)_\infty}{(q^{2x};q)_\infty}(1-q^2)^{1-2x} = (1+q)^{1-2x}\Gamma_q(2x) \end{aligned}$$

and hence

$$\Gamma_q(2x)\Gamma_{q^2}\left(\frac{1}{2}\right) = (1+q)^{2x-1}\Gamma_{q^2}(x)\Gamma_{q^2}\left(x+\frac{1}{2}\right). \tag{1.10.9}$$

Similarly, it can be shown that the Gauss multiplication formula

$$\Gamma(nx)(2\pi)^{(n-1)/2} = n^{nx-\frac{1}{2}}\Gamma(x)\Gamma\left(x+\frac{1}{n}\right)\cdots\Gamma\left(x+\frac{n-1}{n}\right) \tag{1.10.10}$$

has a q-analogue of the form

$$\begin{aligned} &\Gamma_q(nx)\Gamma_r\left(\frac{1}{n}\right)\Gamma_r\left(\frac{2}{n}\right)\cdots\Gamma_r\left(\frac{n-1}{n}\right) \\ &= (1+q+\cdots+q^{n-1})^{nx-1}\Gamma_r(x)\Gamma_r\left(x+\frac{1}{n}\right)\cdots\Gamma_r\left(x+\frac{n-1}{n}\right) \end{aligned} \tag{1.10.11}$$

with $r = q^n$; see Jackson [1904e, 1905d]. The q-gamma function for $q > 1$ is considered in Exercise 1.23. For other interesting properties of the q-gamma function see Askey [1978] and Moak [1980a,b] and Ismail, Lorch and Muldoon [1986].

Since the *beta function* is defined by

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \tag{1.10.12}$$

it is natural to define the *q-beta function* by

$$B_q(x,y) = \frac{\Gamma_q(x)\Gamma_q(y)}{\Gamma_q(x+y)}, \tag{1.10.13}$$

which tends to $B(x,y)$ as $q \to 1^-$. By (1.10.1) and (1.3.2),

$$B_q(x,y) = (1-q)\frac{(q, q^{x+y}; q)_\infty}{(q^x, q^y; q)_\infty}$$

$$= (1-q)\frac{(q;q)_\infty}{(q^y;q)_\infty}\sum_{n=0}^{\infty}\frac{(q^y;q)_n}{(q;q)_n}q^{nx}$$

$$= (1-q)\sum_{n=0}^{\infty}\frac{(q^{n+1};q)_\infty}{(q^{n+y};q)_\infty}q^{nx}, \quad \mathrm{Re}\, x,\ \mathrm{Re}\, y > 0. \quad (1.10.14)$$

This series expansion will be used in the next section to derive a q-integral representation for $B_q(x,y)$.

1.11 The q-integral

Thomae [1869, 1870] and Jackson [1910c, 1951] introduced the *q-integral*

$$\int_0^1 f(t)\, d_qt = (1-q)\sum_{n=0}^{\infty} f(q^n)q^n \quad (1.11.1)$$

and Jackson gave the more general definition

$$\int_a^b f(t)\, d_qt = \int_0^b f(t)\, d_qt - \int_0^a f(t)\, d_qt, \quad (1.11.2)$$

where

$$\int_0^a f(t)\, d_qt = a(1-q)\sum_{n=0}^{\infty} f(aq^n)q^n. \quad (1.11.3)$$

Jackson also defined an integral on $(0,\infty)$ by

$$\int_0^\infty f(t)\, d_qt = (1-q)\sum_{n=-\infty}^{\infty} f(q^n)q^n. \quad (1.11.4)$$

The *bilateral q-integral* is defined by

$$\int_{-\infty}^\infty f(t)\, d_qt = (1-q)\sum_{n=-\infty}^{\infty} [f(q^n) + f(-q^n)]\, q^n. \quad (1.11.5)$$

If f is continuous on $[0,a]$, then it is easily seen that

$$\lim_{q\to 1}\int_0^a f(t)\, d_qt = \int_0^a f(t)\, dt \quad (1.11.6)$$

and that a similar limit holds for (1.11.4) and (1.11.5) when f is suitably restricted. By (1.11.1), it follows from (1.10.14) that

$$B_q(x,y) = \int_0^1 t^{x-1}\frac{(tq;q)_\infty}{(tq^y;q)_\infty}\, d_qt, \ \mathrm{Re}\, x > 0, \quad y \neq 0, -1, -2, \ldots, \quad (1.11.7)$$

which clearly approaches the beta function integral

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\, dt, \quad \mathrm{Re}\, x,\ \mathrm{Re}\, y > 0, \quad (1.11.8)$$

as $q \to 1^-$. Thomae [1869] rewrote Heine's formula (1.4.1) in the q-integral form

$$ {}_2\phi_1(q^a, q^b; q^c; q, z) = \frac{\Gamma_q(c)}{\Gamma_q(b)\Gamma_q(c-b)} \int_0^1 t^{b-1} \frac{(tzq^a, tq; q)_\infty}{(tz, tq^{c-b}; q)_\infty} \, d_q t, \tag{1.11.9} $$

which is a q-analogue of Euler's integral representation

$$ {}_2F_1(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a} \, dt, \tag{1.11.10} $$

where $|\arg(1-z)| < \pi$ and $\mathrm{Re}\, c > \mathrm{Re}\, b > 0$.

The q-integral notation is, as we shall see later, quite useful in simplifying and manipulating various formulas involving sums of series.

Exercises

1.1 Verify the identities (1.2.30)–(1.2.40), and show that

(i) $$(aq^{-n}; q)_n = (q/a; q)_n \left(-\frac{a}{q}\right)^n q^{-\binom{n}{2}},$$

(ii) $$(aq^{-k-n}; q)_n = \frac{(q/a; q)_{n+k}}{(q/a; q)_k} \left(-\frac{a}{q}\right)^n q^{-nk-\binom{n}{2}},$$

(iii) $$\frac{(qa^{\frac{1}{2}}, -qa^{\frac{1}{2}}; q)_n}{(a^{\frac{1}{2}}, -a^{\frac{1}{2}}; q)_n} = \frac{1-aq^{2n}}{1-a},$$

(iv) $$(a; q)_{2n} = (a^{\frac{1}{2}}, -a^{\frac{1}{2}}, (aq)^{\frac{1}{2}}, -(aq)^{\frac{1}{2}}; q)_n,$$

(v) $$(a; q)_n (q/a; q)_{-n} = (-a)^n q^{\binom{n}{2}},$$

(vi) $$(q, -q, -q^2; q^2)_\infty = 1.$$

1.2 The *q-binomial coefficient* is defined by

$$ \begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q; q)_n}{(q; q)_k (q; q)_{n-k}} $$

for $k = 0, 1, \ldots, n$, and by

$$ \begin{bmatrix} \alpha \\ \beta \end{bmatrix}_q = \frac{(q^{\beta+1}, q^{\alpha-\beta+1}; q)_\infty}{(q, q^{\alpha+1}; q)_\infty} = \frac{\Gamma_q(\alpha+1)}{\Gamma_q(\beta+1)\Gamma_q(\alpha-\beta+1)} $$

for complex α and β when $|q| < 1$. Verify that

(i) $$\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{bmatrix} n \\ n-k \end{bmatrix}_q,$$

(ii) $$\begin{bmatrix} \alpha \\ k \end{bmatrix}_q = \frac{(q^{-\alpha}; q)_k}{(q; q)_k} (-q^\alpha)^k q^{-\binom{k}{2}},$$

(iii) $$\begin{bmatrix} k+\alpha \\ k \end{bmatrix}_q = \frac{(q^{\alpha+1}; q)_k}{(q; q)_k},$$

(iv) $$\begin{bmatrix} -\alpha \\ k \end{bmatrix}_q = \begin{bmatrix} \alpha+k-1 \\ k \end{bmatrix}_q (-q^{-\alpha})^k q^{-\binom{k}{2}},$$

(v) $$\begin{bmatrix} \alpha+1 \\ k \end{bmatrix}_q = \begin{bmatrix} \alpha \\ k \end{bmatrix}_q q^k + \begin{bmatrix} \alpha \\ k-1 \end{bmatrix}_q = \begin{bmatrix} \alpha \\ k \end{bmatrix}_q + \begin{bmatrix} \alpha \\ k-1 \end{bmatrix}_q q^{\alpha+1-k},$$

(vi) $$(z;q)_n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q (-z)^k q^{\binom{k}{2}},$$

when k and n are nonnegative integers.

1.3 (i) Show that the *binomial theorem*

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$$

where $n = 0, 1, \ldots,$ has a q-analogue of the form

$$\begin{aligned}(ab;q)_n &= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q b^k (a;q)_k (b;q)_{n-k} \\ &= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q a^{n-k} (a;q)_k (b;q)_{n-k}.\end{aligned}$$

(ii) Extend the above formula to the *q-multinomial theorem*

$$\begin{aligned}&(a_1 a_2 \cdots a_{m+1};q)_n \\ &= \sum_{\substack{0 \le k_1, \ldots, 0 \le k_m \\ k_1 + \cdots + k_m \le n}} \begin{bmatrix} n \\ k_1, \ldots, k_m \end{bmatrix}_q a_2^{k_1} a_3^{k_1+k_2} \cdots a_{m+1}^{k_1+k_2+\cdots+k_m} \\ &\quad \times (a_1;q)_{k_1} (a_2;q)_{k_2} \cdots (a_m;q)_{k_m} (a_{m+1};q)_{n-(k_1+\cdots+k_m)},\end{aligned}$$

where $m = 1, 2, \ldots,\ n = 0, 1, \ldots,$ and

$$\begin{bmatrix} n \\ k_1, \ldots, k_m \end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_{k_1} \cdots (q;q)_{k_m} (q;q)_{n-(k_1+\cdots+k_m)}}$$

is the *q-multinomial coefficient.*

1.4 (i) Prove the inversion formula

$$\begin{aligned}&{}_r\phi_s \left[\begin{matrix} a_1, \ldots, a_r \\ b_1, \ldots, b_s \end{matrix} ; q, z \right] \\ &= \sum_{n=0}^{\infty} \frac{(a_1^{-1}, \ldots, a_r^{-1}; q^{-1})_n}{(q^{-1}, b_1^{-1}, \ldots, b_s^{-1}; q^{-1})_n} \left(\frac{a_1 \cdots a_r z}{b_1 \cdots b_s q} \right)^n.\end{aligned}$$

(ii) By reversing the order of summation, show that

$$\begin{aligned}&{}_{r+1}\phi_s \left[\begin{matrix} a_1, \ldots, a_r, q^{-n} \\ b_1, \ldots, b_s \end{matrix} ; q, z \right] \\ &= \frac{(a_1, \ldots, a_r; q)_n}{(b_1 \ldots, b_s; q)_n} \left(\frac{z}{q} \right)^n \left((-1)^n q^{\binom{n}{2}} \right)^{s-r-1} \\ &\quad \times \sum_{k=0}^{n} \frac{(q^{1-n}/b_1, \ldots, q^{1-n}/b_s, q^{-n}; q)_k}{(q, q^{1-n}/a_1, \ldots, q^{1-n}/a_r; q)_k} \left(\frac{b_1 \cdots b_s}{a_1 \cdots a_r} \frac{q^{n+1}}{z} \right)^k\end{aligned}$$

when $n = 0, 1, \ldots$.

(iii) Show that

$$
\begin{aligned}
&{}_{r+1}\phi_r\left[\begin{matrix} a_1, \ldots, a_{r+1} \\ b_1, \ldots, b_r \end{matrix}; q, q^z\right] \\
&= \frac{(a_1, \ldots, a_{r+1}; q)_\infty}{(1-q)(q, b_1, \ldots, b_r; q)_\infty} \int_0^1 t^{z-1} \frac{(qt, b_1 t, \ldots, b_r t; q)_\infty}{(a_1 t, \ldots, a_{r+1} t; q)_\infty} d_q t,
\end{aligned}
$$

when $0 < q < 1$, Re $z > 0$, and the series on the left side does not terminate.

1.5 Show that

$$
\frac{(c, bq^n; q)_m}{(b; q)_m} = \frac{(b/c; q)_n}{(b; q)_n} \sum_{k=0}^{n} \frac{(q^{-n}, c; q)_k q^k}{(q, cq^{1-n}/b; q)_k} (cq^k; q)_m.
$$

1.6 Prove the summation formulas

(i) ${}_2\phi_1(q^{-n}, q^{1-n}; qb^2; q^2, q^2) = \dfrac{(b^2; q^2)_n}{(b^2; q)_n} q^{-\binom{n}{2}}$,

(ii) ${}_1\phi_1(a; c; q, c/a) = \dfrac{(c/a; q)_\infty}{(c; q)_\infty}$,

(iii) ${}_2\phi_0(a, q^{-n}; -; q, q^n/a) = a^{-n}$,

(iv) $\displaystyle\sum_{n=0}^{\infty} \frac{q^{n^2-n}}{(q; q)_n^2} = \frac{2}{(q; q)_\infty}$,

(v) ${}_1\phi_0(a; -; p, z) = \dfrac{(zp^{-1}; p^{-1})_\infty}{(azp^{-1}; p^{-1})_\infty}$, $\quad |p| > 1,\ |azp^{-1}| < 1$,

(vi) ${}_2\phi_1(a, b; c; p, p) = \dfrac{(a/c, b/c; p^{-1})_\infty}{(1/c, ab/c; p^{-1})_\infty}$, $\quad |p| > 1$.

1.7 Show that, for $|z| < 1$,

$$
{}_2\phi_1(a^2, aq; a; q, z) = (1 + az) \frac{(a^2 qz; q)_\infty}{(z; q)_\infty}.
$$

1.8 Show that, when $|a| < 1$ and $|bq/a^2| < 1$,

$$
\begin{aligned}
&{}_2\phi_1(a^2, a^2/b; b; q^2, bq/a^2) \\
&= \frac{(a^2, q; q^2)_\infty}{2(b, bq/a^2; q^2)_\infty} \left[\frac{(b/a; q)_\infty}{(a; q)_\infty} + \frac{(-b/a; q)_\infty}{(-a; q)_\infty}\right].
\end{aligned}
$$

(Andrews and Askey [1977])

1.9 Let $\phi(a, b, c)$ denote the series ${}_2\phi_1(a, b; c; q, z)$. Verify Heine's [1847] *q-contiguous relations*:

(i) $\phi(a, b, cq^{-1}) - \phi(a, b, c) = cz \dfrac{(1-a)(1-b)}{(q-c)(1-c)} \phi(aq, bq, cq)$,

(ii) $\phi(aq, b, c) - \phi(a, b, c) = az \dfrac{1-b}{1-c} \phi(aq, bq, cq)$,

(iii) $\phi(aq, b, cq) - \phi(a, b, c) = az\dfrac{(1-b)(1-c/a)}{(1-c)(1-cq)}\phi(aq, bq, cq^2)$,

(iv) $\phi(aq, bq^{-1}, c) - \phi(a, b, c) = az\dfrac{(1-b/aq)}{1-c}\phi(aq, b, cq)$.

1.10 Denoting ${}_2\phi_1(a, b; c; q, z)$, ${}_2\phi_1(aq^{\pm 1}, b; c, q, z)$, ${}_2\phi_1(a, bq^{\pm 1}; c; q, z)$ and ${}_2\phi_1(a, b; cq^{\pm 1}; q, z)$ by $\phi, \phi(aq^{\pm 1}), \phi(bq^{\pm 1})$ and $\phi(cq^{\pm 1})$, respectively, show that

(i) $b(1-a)\phi(aq) - a(1-b)\phi(bq) = (b-a)\phi$,
(ii) $a(1-b/c)\phi(bq^{-1}) - b(1-a/c)\phi(aq^{-1}) = (a-b)(1-abz/cq)\phi$,
(iii) $q(1-a/c)\phi(aq^{-1}) + (1-a)(1-abz/c)\phi(aq)$
$= [1+q-a-aq/c+a^2z(1-b/a)/c]\phi$,
(iv) $(1-c)(q-c)(abz-c)\phi(cq^{-1}) + (c-a)(c-b)z\phi(cq)$
$= (c-1)[c(q-c)+(ca+cb-ab-abq)z]\phi$.
(Heine [1847])

1.11 Let $g(\theta; \lambda, \mu, \nu) = (\lambda e^{i\theta}, \mu\nu; q)_\infty\, {}_2\phi_1(\mu e^{-i\theta}, \nu e^{-i\theta}; \mu\nu; q, \lambda e^{i\theta})$. Prove that $g(\theta; \lambda, \mu, \nu)$ is symmetric in λ, μ, ν and is even in θ.

1.12 Let $\mathcal{D}_q$ be the *q-derivative operator* defined for fixed q by

$$\mathcal{D}_q f(z) = \frac{f(z) - f(qz)}{(1-q)z},$$

and let $\mathcal{D}_q^n u = \mathcal{D}_q(D_q^{n-1}u)$ for $n = 1, 2, \ldots$. Show that

(i) $\lim\limits_{q\to 1} \mathcal{D}_q f(z) = \dfrac{d}{dz}f(z)$ if f is differentiable at z,

(ii) $\mathcal{D}_q^n\, {}_2\phi_1(a, b; c; q, z) = \dfrac{(a, b; q)_n}{(c; q)_n(1-q)^n}\, {}_2\phi_1(aq^n, bq^n; cq^n; q, z)$,

(iii) $\mathcal{D}_q^n\left\{\dfrac{(z; q)_\infty}{(abz/c; q)_\infty}\, {}_2\phi_1(a, b; c; q, z)\right\}$
$= \dfrac{(c/a, c/b; q)_n}{(c; q)_n(1-q)^n}\left(\dfrac{ab}{c}\right)^n \dfrac{(zq^n; q)_\infty}{(abz/c; q)_\infty}\, {}_2\phi_1(a, b; cq^n; q, zq^n)$.

(iv) Prove the q-Leibniz formula

$$\mathcal{D}_q^n[f(z)g(z)] = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \mathcal{D}_q^{n-k} f(zq^k)\mathcal{D}_q^k g(z).$$

1.13 Show that $u(z) = {}_2\phi_1(a, b; c; q, z)$ satisfies (for $|z| < 1$ and in the formal power series sense) the second order q-differential equation

$$z(c - abqz)\mathcal{D}_q^2 u + \left[\frac{1-c}{1-q} + \frac{(1-a)(1-b)-(1-abq)}{1-q}z\right]\mathcal{D}_q u$$
$$- \frac{(1-a)(1-b)}{(1-q)^2}u = 0,$$

where $\mathcal{D}_q$ is defined as in Ex. 1.12. By replacing a, b, c, respectively, by q^a, q^b, q^c and then letting $q \to 1^-$ show that the above equation tends to the second order differential equation

$$z(1-z)v'' + [c - (a+b+1)z]v' - abv = 0$$

for the hypergeometric function $v(z) = {}_2F_1(a, b; c; z)$, where $|z| < 1$. (Heine [1847])

1.14 Let $|x| < 1$ and let $e_q(x)$ and $E_q(x)$ be as defined in §1.3. Define

$$\sin_q(x) = \frac{e_q(ix) - e_q(-ix)}{2i} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(q;q)_{2n+1}},$$

$$\cos_q(x) = \frac{e_q(ix) + e_q(-ix)}{2} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(q;q)_{2n}}.$$

Also define

$$\mathrm{Sin}_q(x) = \frac{E_q(ix) - E_q(-ix)}{2i}, \quad \mathrm{Cos}_q(x) = \frac{E_q(ix) + E_q(-ix)}{2}.$$

Show that

(i) $$e_q(ix) = \cos_q(x) + i\sin_q(x),$$

(ii) $$E_q(ix) = \mathrm{Cos}_q(x) + i\mathrm{Sin}_q(x),$$

(iii) $$\sin_q(x)\mathrm{Sin}_q(x) + \cos_q(x)\mathrm{Cos}_q(x) = 1,$$

(iv) $$\sin_q(x)\mathrm{Cos}_q(x) - \mathrm{Sin}_q(x)\cos_q(x) = 0.$$

For these identities and other identities involving q-analogues of $\sin x$ and $\cos x$, see Jackson [1904a] and Hahn [1949c].

1.15 Prove the transformation formulas

(i) $${}_2\phi_1\left[\begin{matrix} q^{-n}, b \\ c \end{matrix}; q, z\right] = \frac{(bzq^{-n}/c;q)_\infty}{(bz/c;q)_\infty}\, {}_3\phi_2\left[\begin{matrix} q^{-n}, c/b, 0 \\ c, cq/bz \end{matrix}; q, q\right],$$

(ii) $${}_2\phi_1\left[\begin{matrix} q^{-n}, b \\ c \end{matrix}; q, z\right] = \frac{(c/b;q)_n}{(c;q)_n} b^n\, {}_3\phi_1\left[\begin{matrix} q^{-n}, b, q/z \\ bq^{1-n}/c \end{matrix}; q, z/c\right],$$

(iii) $${}_2\phi_1\left[\begin{matrix} q^{-n}, b \\ c \end{matrix}; q, z\right] = \frac{(c/b;q)_n}{(c;q)_n}\, {}_3\phi_2\left[\begin{matrix} q^{-n}, b, bzq^{-n}/c \\ bq^{1-n}/c, \ 0 \end{matrix}; q, q\right].$$

(See Jackson [1905a, 1927])

1.16 Show that

$$\sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n} q^{n(n+1)/2} = (-q;q)_\infty (aq;q^2)_\infty.$$

1.17 Show that

$$\sum_{k=0}^{n} \frac{(a,b;q)_k}{(q;q)_k} (-ab)^{n-k} q^{(n-k)(n+k-1)/2}$$

$$= (a;q)_{n+1} \sum_{k=0}^{n} \frac{(-b)^k q^{\binom{k}{2}}}{(q;q)_k (q;q)_{n-k} (1 - aq^{n-k})}. \qquad \text{(Carlitz [1974])}$$

1.18 Show that

(i) $$(c;q)_\infty\ {}_1\phi_1(a;c;q,z) = (z;q)_\infty\ {}_1\phi_1(az/c;z;q,c),$$
and deduce that ${}_1\phi_1(-bq;0;q,-q) = (-bq^2;q^2)_\infty/(q;q^2)_\infty$,

(ii) $$(z;q)_\infty\ {}_2\phi_1(a,0;c;q,z) = (az;q)_\infty\ {}_1\phi_2(a;c,az;q,cz),$$

(iii) $$\sum_{n=0}^{\infty} \frac{(a;q)_n}{(q,a^2;q)_n} q^{\binom{n}{2}} (at/z)^n\ {}_2\phi_1(q^{-n},a;q^{1-n}/a;q,qz^2/a)$$
$$= (-zt;q)_\infty\ {}_2\phi_1(a,a/z^2;a^2;q,-zt), \quad |zt|<1.$$

1.19 Using (1.5.4) show that

(i) $${}_2\phi_2\left[\begin{matrix} a, q/a \\ -q, b \end{matrix}; q, -b\right] = \frac{(ab, bq/a;q^2)_\infty}{(b;q)_\infty},$$

(ii) $${}_2\phi_2\left[\begin{matrix} a^2, b^2 \\ abq^{\frac{1}{2}}, -abq^{\frac{1}{2}} \end{matrix}; q, -q\right] = \frac{(a^2q, b^2q;q^2)_\infty}{(q, a^2b^2q;q^2)_\infty}.$$

(Andrews [1973])

1.20 Prove that if Re $x>0$ and $0<q<1$, then

(i) $$\Gamma_q(x) = (q;q)_\infty (1-q)^{1-x} \sum_{n=0}^{\infty} \frac{q^{nx}}{(q;q)_n},$$

(ii) $$\frac{1}{\Gamma_q(x)} = \frac{(1-q)^{x-1}}{(q;q)_\infty} \sum_{n=0}^{\infty} \frac{(-1)^n q^{nx}}{(q;q)_n} q^{\binom{n}{2}}.$$

1.21 For $0<q<1$ and $x>0$, show that

$$\frac{d^2}{dx^2} \log \Gamma_q(x) = (\log q)^2 \sum_{n=0}^{\infty} \frac{q^{n+x}}{(1-q^{n+x})^2},$$

which proves that $\log \Gamma_q(x)$ is convex for $x>0$ when $0<q<1$.

1.22 Conversely, prove that if $f(x)$ is a positive function defined on $(0,\infty)$which satisfies

$$f(x+1) = \frac{1-q^x}{1-q} f(x) \text{ for some } q, 0<q<1,$$
$$f(1) = 1,$$

and $\log f(x)$ is convex for $x>0$, then $f(x) = \Gamma_q(x)$. This is Askey's [1978] q-analogue of the Bohr-Mollerup [1922] theorem for $\Gamma(x)$. For two extensions to the $q>1$ case (with $\Gamma_q(x)$ defined as in the next exercise), see Moak [1980b].

1.23 For $q>1$ the *q-gamma function* is defined by

$$\Gamma_q(x) = \frac{(q^{-1};q^{-1})_\infty}{(q^{-x};q^{-1})_\infty} (q-1)^{1-x} q^{x(x-1)/2}.$$

Show that this function also satisfies the functional equation (1.10.5) and that $\Gamma_q(x) \to \Gamma(x)$ as $q \to 1^+$. Show that for $q > 1$ the residue of $\Gamma_q(x)$ at $x = -n$ is

$$\frac{(q-1)^{n+1} q^{\binom{n+1}{2}}}{(q;q)_n \log q}.$$

1.24 Jackson [1905a,b,e] gave the following q-analogues of Bessel functions:

$$J_\nu^{(1)}(x;q) = \frac{(q^{\nu+1};q)_\infty}{(q;q)_\infty}(x/2)^\nu \, {}_2\phi_1(0,0;q^{\nu+1};q,-x^2/4),$$
$$J_\nu^{(2)}(x;q) = \frac{(q^{\nu+1};q)_\infty}{(q;q)_\infty}(x/2)^\nu \, {}_0\phi_1\left(—;q^{\nu+1};q,-\frac{x^2 q^{\nu+1}}{4}\right),$$
$$J_\nu^{(3)}(x;q) = \frac{(q^{\nu+1};q)_\infty}{(q;q)_\infty}(x/2)^\nu \, {}_1\phi_1(0;q^{\nu+1};q,qx^2/4),$$

where $0 < q < 1$. The above notations for the *q-Bessel functions* are due to Ismail [1981, 1982, 2003c].
Show that

$$J_\nu^{(2)}(x;q) = \left(-x^2/4;q\right)_\infty J_\nu^{(1)}(x;q), \quad |x| < 2, \quad \text{(Hahn [1949c])}$$

and

$$\lim_{q\to 1} J_\nu^{(k)}(x(1-q);q) = J_\nu(x), \qquad k = 1,2,3.$$

1.25 For the q-Bessel functions defined as in Exercise 1.24 prove that

(i) $$q^\nu J_{\nu+1}^{(k)}(x;q) = \frac{2(1-q^\nu)}{x} J_\nu^{(k)}(x;q) - J_{\nu-1}^{(k)}(x;q), \; k = 1,2;$$

(ii) $$J_\nu^{(1)}(xq^{\frac{1}{2}};q) = q^{\nu/2}\left(J_\nu^{(1)}(x;q) + \frac{x}{2} J_{\nu+1}^{(1)}(x;q)\right);$$

(iii) $$J_\nu^{(1)}(xq^{\frac{1}{2}};q) = q^{-\nu/2}\left(J_\nu^{(1)}(x;q) - \frac{x}{2} J_{\nu-1}^{(1)}(x;q)\right).$$

(iv) $$q^{\nu+1} J_{\nu+1}^{(3)}(xq^{1/2};q) = \frac{2(1-q^\nu)}{x} J_\nu^{(3)}(x;q) - J_{\nu-1}^{(3)}(x;q).$$

1.26 (i) Following Ismail [1982], let

$$f_\nu(x) = J_\nu^{(1)}(x;q) J_{-\nu}^{(1)}(xq^{\frac{1}{2}};q) - J_{-\nu}^{(1)}(x;q) J_\nu^{(1)}(xq^{\frac{1}{2}};q).$$

Show that

$$f_\nu(xq^{\frac{1}{2}}) = \left(1 + \frac{x^2}{4}\right) f_\nu(x)$$

and deduce that, for non-integral ν,

$$f_\nu(x) = q^{-\nu/2}(q^\nu, q^{1-\nu};q)_\infty / (q,q,-x^2/4;q)_\infty.$$

(ii) Show that

$$g_\nu(qx) + \left(x^2/4 - q^\nu - q^{-\nu}\right) g_\nu(x) + g_\nu(xq^{-1}) = 0$$

with $g_\nu(x) = J_\nu^{(3)}(xq^{\nu/2}; q^2)$ and deduce that

$$g_\nu(x)g_{-\nu}(xq^{-1}) - g_{-\nu}(x)g_\nu(xq^{-1}) = \frac{(q^{2\nu}, q^{1-2\nu}; q^2)_\infty}{(q^2, q^2; q^2)_\infty} q^{\nu(\nu-1)}.$$

(Ismail [2003c])

1.27 Show that

(i) $$\sum_{n=-\infty}^{\infty} t^n J_n^{(2)}(x;q) = \left(-x^2/4; q\right)_\infty e_q(xt/2) e_q(-x/2t),$$

(ii) $$\sum_{n=-\infty}^{\infty} t^n J_n^{(3)}(x;q) = e_q(xt/2) E_q(-qx/2t).$$

Both of these are q-analogues of the generating function

$$\sum_{n=-\infty}^{\infty} t^n J_n(x) = e^{x(t-t^{-1})/2}.$$

1.28 The *continuous q-Hermite polynomials* are defined in Askey and Ismail [1983] by

$$H_n(x|q) = \sum_{k=0}^{n} \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}} e^{i(n-2k)\theta},$$

where $x = \cos\theta$; see Szegő [1926], Carlitz [1955, 1957a, 1958, 1960] and Rogers [1894, 1917]. Derive the generating function

$$\sum_{n=0}^{\infty} \frac{H_n(x|q)}{(q;q)_n} t^n = \frac{1}{(te^{i\theta}, te^{-i\theta}; q)_\infty}, \quad |t| < 1. \quad \text{(Rogers [1894])}$$

1.29 The *continuous q-ultraspherical polynomials* are defined in Askey and Ismail [1983] by

$$C_n(x;\beta|q) = \sum_{k=0}^{n} \frac{(\beta;q)_k (\beta;q)_{n-k}}{(q;q)_k (q;q)_{n-k}} e^{i(n-2k)\theta},$$

where $x = \cos\theta$. Show that

$$\begin{aligned} C_n(x;\beta|q) &= \frac{(\beta;q)_n}{(q;q)_n} e^{in\theta} {}_2\phi_1 \left[\begin{matrix} q^{-n}, \beta \\ \beta^{-1} q^{1-n} \end{matrix} ; q, q\beta^{-1} e^{-2i\theta} \right] \\ &= \frac{(\beta^2;q)_n}{(q;q)_n} e^{-in\theta} \beta^{-n} {}_3\phi_2 \left[\begin{matrix} q^{-n}, \beta, \beta e^{2i\theta} \\ \beta^2, 0 \end{matrix} ; q, q \right], \\ &\lim_{q\to 1} C_n(x; q^\lambda|q) = C_n^\lambda(x), \end{aligned}$$

and

$$\sum_{n=0}^{\infty} C_n(x;\beta|q) t^n = \frac{(\beta te^{i\theta}, \beta te^{-i\theta}; q)_\infty}{(te^{i\theta}, te^{-i\theta}; q)_\infty}, \ |t| < 1. \quad \text{(Rogers [1895])}$$

1.30 Show that if $m_1, \ldots, m_r$ are nonnegative integers, then

(i)
$$
{}_{r+1}\phi_{r+1}\left[\begin{matrix} b, & b_1 q^{m_1}, \ldots, b_r q^{m_r} \\ & bq, b_1, \ldots, b_r \end{matrix} ; q, q^{1-(m_1+\cdots+m_r)}\right]
= \frac{(q;q)_\infty (b_1/b;q)_{m_1} \cdots (b_r/b;q)_{m_r}}{(bq;q)_\infty (b_1;q)_{m_1} \cdots (b_r;q)_{m_r}} b^{m_1+\cdots+m_r},
$$

(ii)
$$
{}_r\phi_r\left[\begin{matrix} b_1 q^{m_1}, \ldots, b_r q^{m_r} \\ b_1, \ldots, b_r \end{matrix} ; q, q^{-(m_1+\cdots+m_r)}\right] = 0,
$$

(iii)
$$
{}_r\phi_r\left[\begin{matrix} b_1 q^{m_1}, \ldots, b_r q^{m_r} \\ b_1, \ldots, b_r \end{matrix} ; q, q^{1-(m_1+\cdots+m_r)}\right]
= \frac{(-1)^{m_1+\cdots+m_r} (q;q)_\infty b_1^{m_1} \cdots b_r^{m_r}}{(b_1;q)_{m_1} \cdots (b_r;q)_{m_r}} q^{\binom{m_1}{2}+\cdots+\binom{m_r}{2}}.
$$

(Gasper [1981a])

1.31 Let Δ_b denote the *q-difference operator* defined for a fixed q by

$$\Delta_b f(z) = bf(qz) - f(z).$$

Then Δ_1 is the Δ operator defined in (1.3.20). Show that

$$\Delta_b x^n = (bq^n - 1)x^n$$

and, if

$$v_n(z) = \frac{(a_1, \ldots, a_r; q)_n}{(q, b_1, \ldots, b_s; q)_n} (-1)^{(1+s-r)n} q^{(1+s-r)n(n-1)/2} z^n,$$

then

$$
\begin{aligned}
&(\Delta\Delta_{b_1/q}\Delta_{b_2/q} \cdots \Delta_{b_s/q}) v_n(z) \\
&= z(\Delta_{a_1}\Delta_{a_2} \cdots \Delta_{a_r}) v_{n-1}(zq^{1+s-r}), \quad n = 1, 2, \ldots .
\end{aligned}
$$

Use this to show that the basic hypergeometric series

$$v(z) = {}_r\phi_s(a_1, \ldots, a_r; b_1, \ldots, b_s; q, z)$$

satisfies (in the sense of formal power series) the q-difference equation

$$(\Delta\Delta_{b_1/q}\Delta_{b_2/q} \cdots \Delta_{b_s/q}) v(z) = z(\Delta_{a_1} \cdots \Delta_{a_r}) v(zq^{1+s-r}).$$

This is a q-analogue of the formal differential equation for generalized hypergeometric series given, e.g. in Henrici [1974, Theorem (1.5)] and Slater [1966, (2.1.2.1)]. Also see Jackson [1910d, (15)].

1.32 The *little q-Jacobi polynomials* are defined by

$$p_n(x; a, b; q) = {}_2\phi_1(q^{-n}, abq^{n+1}; aq; q, qx).$$

Show that these polynomials satisfy the orthogonality relation

$$
\sum_{j=0}^{\infty} \frac{(bq;q)_j}{(q;q)_j} (aq)^j p_n(q^j; a, b; q) p_m(q^j; a, b; q)
= \begin{cases} 0, & \text{if } m \neq n, \\ \dfrac{(q, bq; q)_n (1 - abq)(aq)^n}{(aq, abq; q)_n (1 - abq^{2n+1})} \dfrac{(abq^2; q)_\infty}{(aq; q)_\infty}, & \text{if } m = n. \end{cases}
$$

(Andrews and Askey [1977])

1.33 Show for the above little q-Jacobi polynomials that the formula

$$p_n(x;c,d;q) = \sum_{k=0}^{n} a_{k,n} p_k(x;a,b;q)$$

holds with

$$a_{k,n} = (-1)^k q^{\binom{k+1}{2}} \frac{(q^{-n}, aq, cdq^{n+1};q)_k}{(q, cq, abq^{k+1};q)_k} \, {}_3\phi_2 \left[\begin{matrix} q^{k-n}, cdq^{n+k+1}, aq^{k+1} \\ cq^{k+1}, abq^{2k+2} \end{matrix} ;q,q \right].$$

(Andrews and Askey [1977])

1.34 (i) If $m, m_1, m_2, \ldots, m_r$ are arbitrary nonnegative integers and $|a^{-1}q^{m+1-(m_1+\cdots+m_r)}| < 1$, show that

$$\begin{aligned} & {}_{r+2}\phi_{r+1} \left[\begin{matrix} a, b, b_1 q^{m_1}, \ldots, b_r q^{m_r} \\ bq^{1+m}, b_1, \ldots, b_r \end{matrix} ;q, a^{-1} q^{m+1-(m_1+\cdots+m_r)} \right] \\ &= \frac{(q, bq/a;q)_\infty (bq;q)_m (b_1/b;q)_{m_1} \cdots (b_r/b;q)_{m_r}}{(bq, q/a;q)_\infty (q;q)_m (b_1;q)_{m_1} \cdots (b_r;q)_{m_r}} b^{m_1+\cdots+m_r-m} \\ &\times {}_{r+2}\phi_{r+1} \left[\begin{matrix} q^{-m}, b, bq/b_1, \ldots, bq/b_r \\ bq/a, bq^{1-m_1}/b_1, \ldots, bq^{1-m_r}/b_r \end{matrix} ;q,q \right]; \end{aligned}$$

(ii) if $m_1, m_2, \ldots, m_r$ are nonnegative integers and $|a^{-1}q^{1-(m_1+\cdots+m_r)}| < 1$, $|cq| < 1$, show that

$$\begin{aligned} & {}_{r+2}\phi_{r+1} \left[\begin{matrix} a, b, b_1 q^{m_1}, \ldots, b_r q^{m_r} \\ bcq, b_1, \ldots, b_r \end{matrix} ;q, a^{-1} q^{1-(m_1+\cdots+m_r)} \right] \\ &= \frac{(bq/a, cq;q)_\infty}{(bcq, q/a;q)_\infty} \frac{(b_1/b;q)_{m_1} \cdots (b_r/b;q)_{m_r}}{(b_1;q)_{m_1} \cdots (b_r;q)_{m_r}} b^{m_1+\cdots+m_r} \\ &\times {}_{r+2}\phi_{r+1} \left[\begin{matrix} c^{-1}, b, bq/b_1, \ldots, bq/b_r \\ bq/a, bq^{1-m_1}/b_1, \ldots, bq^{1-m_r}/b_r \end{matrix} ;q, cq \right]. \end{aligned}$$

(Gasper [1981a])

1.35 Use Ex. 1.2(v) to prove that if x and y are indeterminates such that $xy = qyx$, q commutes with x and y, and the associative law holds, then

$$(x+y)^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q y^k x^{n-k} = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{q^{-1}} x^k y^{n-k}.$$

(See Cigler [1979], Feinsilver [1982], Koornwinder [1989], Potter[1950], Schützenberger [1953], and Yang [1991]).

1.36 Verify that if x and y are indeterminates satisfying the conditions in Ex. 1.35, then

(i) $\quad e_q(y)e_q(x) = e_q(x+y), \quad e_q(x)e_q(y) = e_q(x+y-yx);$

(ii) $\quad E_q(x)E_q(y) = E_q(x+y), \quad E_q(y)E_q(x) = E_q(x+y+yx).$

(Fairlie and Wu [1997]; Koornwinder [1997], where q-exponentials with q-Heisenberg relations and other relations are also considered.)

1.37 Show that

$$\mathcal{E}_q(z;\alpha) = \frac{(\alpha^2;q^2)_\infty}{(q\alpha^2;q^2)_\infty}\left\{ {}_2\phi_1\left[\begin{matrix}-qe^{2i\theta}, -qe^{-2i\theta} \\ q\end{matrix}; q^2, \alpha^2\right] \right.$$
$$\left. + \frac{2q^{1/4}}{1-q}\alpha\cos\theta\, {}_2\phi_1\left[\begin{matrix}-q^2e^{2i\theta}, -q^2e^{-2i\theta} \\ q^3\end{matrix}; q^2, \alpha^2\right]\right\}$$

with $z = \cos\theta$.

1.38 Extend Jacobi's triple product identity to the transformation formula

$$1 + \sum_{n=1}^{\infty}(-1)^n q^{\binom{n}{2}}(a^n + b^n) = (q,a,b;q)_\infty \sum_{n=0}^{\infty} \frac{(ab/q;q)_{2n} q^n}{(q,a,b,ab;q)_n}.$$

Deduce that

$$1 + 2\sum_{n=1}^{\infty} a^n q^{2n^2} = (q;q)_\infty (aq;q^2)_\infty \sum_{n=0}^{\infty} \frac{(-a;q)_{2n} q^n}{(q,-aq;q)_n (aq;q^2)_n}.$$

(Warnaar [2003a])

Notes

§§1.1 and 1.2 For additional material on hypergeometric series and orthogonal polynomials see, e.g., the books by Erdélyi [1953], Rainville [1960], Szegő [1975], Whittaker and Watson [1965], Agarwal [1963], Carlson [1977], T.S. Chihara [1978], Henrici [1974], Luke [1969], Miller [1968], Nikiforov and Uvarov [1988], Vilenkin [1968], and Watson [1952]. Some techniques for using symbolic computer algebraic systems such as Mathematica, Maple, and Macsyma to derive formulas containing hypergeometric and basic hypergeometric series are discussed in Gasper [1990]. Also see Andrews [1984d, 1986, 1987b], Andrews, Crippa and Simon [1997], Andrews and Knopfmacher [2001], Andrews, Knopfmacher, Paule and Zimmermann [2001], Andrews, Paule and Riese [2001a,b], Askey [1989f, 1990], Askey, Koepf and Koornwinder [1999], Böing and Koepf [1999], Garoufalidis [2003], Garoufalidis, Le and Zeilberger [2003], Garvan [1999], Garvan and Gonnet [1992], Gosper [2001], Gosper and Suslov [2000], Koepf [1998], Koornwinder [1991b, 1993a, 1998], Krattenthaler [1995b], Paule and Riese [1997], Petkovsek, Wilf and Zeilberger [1996], Riese [2003], Sills [2003c], Wilf and Zeilberger [1990], and Zeilberger [1990b].

§§1.3–1.5 The q-binomial theorem was also derived in Jacobi [1846], along with the q-Vandermonde formula. Bijective proofs of the q-binomial theorem, Heine's ${}_2\phi_1$ transformation and q-analogue of Gauss' summation formula, the q-Saalschütz formula, and of other formulas are presented in Joichi and Stanton [1987]. Rahman and Suslov [1996a] used the method of first order linear difference equations to prove the q-binomial and q-Gauss formulas. Bender [1971] used partitions to derive an extension of the q-Vandermonde sum in the form of a generalized q-binomial Vandermonde convolution. The even and odd parts of the infinite series on the right side of (1.3.33) appeared in Atakishiyev and Suslov [1992a], but without any explicit reference to the q-exponential function. Also see Suslov [1998–2003] and the q-convolutions in Carnovale [2002], Carnovale and Koornwinder [2000], and Rogov [2000].

§1.6 Other proofs of Jacobi's triple product identity and/or applications of it are presented in Adiga *et al.* [1985], Alladi and Berkovich [2003], Andrews [1965], Cheema [1964], Ewell [1981], Gustafson [1989], Joichi and Stanton [1989], Kac [1978, 1985], Lepowsky and Milne [1978], Lewis [1984], Macdonald [1972], Menon [1965], Milne [1985a], Sudler [1966], Sylvester [1882], and Wright [1965]. Concerning theta functions, see Adiga *et al.* [1985], Askey [1989c], Bellman [1961], and Jensen's use of theta functions in Pólya [1927] to derive necessary and sufficient conditions for the Riemann hypothesis to hold.

§1.7 Some applications of the q-Saalschütz formula are contained in Carlitz [1969b] and Wright [1968].

§1.9 Formulas (1.9.3) and (1.9.8) were rediscovered by Gustafson [1987a, Theorems 3.15 and 3.18] while working on multivariable orthogonal polynomials.

§1.11 Also see Jackson [1917, 1951] and, for fractional q-integrals and q-derivatives, Al-Salam [1966] and Agarwal [1969b]. Toeplitz [1963, pp. 53–55] pointed out that around 1650 Fermat used a q-integral type Riemann sum to evaluate the integral of x^k on the interval $[0, b]$. Al-Salam and Ismail [1994] evaluated a q-beta integral on the unit circle and found corresponding systems of biorthogonal rational functions.

Ex. 1.2 The q-binomial coefficient $\begin{bmatrix} n \\ k \end{bmatrix}_q$, which is also called the Gaussian binomial coefficient, counts the number of k dimensional subspaces of an n dimensional vector space over a field $GF(q)$, q a prime power (Goldman and Rota [1970]), and it is the generating function, in powers of q, for partitions into at most k parts not exceeding $n-k$ (Sylvester [1882]). It arises in such diverse fields as analysis, computer programming, geometry, number theory, physics, and statistics. See, e.g., Aigner [1979], Andrews [1971a, 1976], M. Baker and Coon [1970], Baxter and Pearce [1983], Berman and Fryer [1972], Dowling [1973], Dunkl [1981], Garvan and Stanton [1990], Handa and Mohanty [1980], Ihrig and Ismail [1981], Jimbo [1985, 1986], van Kampen [1961], Kendall and Stuart [1979, §31.25], Knuth [1971, 1973], Pólya [1970], Pólya and Alexanderson [1970], Szegő [1975, §2.7], and Zaslavsky [1987]. Sylvester [1878] used the invariant theory that he and Cayley developed to prove that the coefficients of the Gaussian polynomial $\begin{bmatrix} n \\ k \end{bmatrix}_q = \Sigma a_j q^j$ are unimodal. A constructive proof was recently given by O'Hara [1990]. Also see Bressoud [1992] and Zeilberger [1989a,b, 1990b]. The unimodality of the sequence $\left(\begin{bmatrix} n \\ k \end{bmatrix}_q : k = 0, 1, \ldots, n\right)$ is explicitly displayed in Aigner [1979, Proposition 3.13], and Macdonald [1995, Example 4 on p. 137].

Ex. 1.3 Cigler [1979] derived an operator form of the q-binomial theorem. MacMahon [1916, Arts. 105–107] showed that if a multiset is permuted, then the generating function for inversions is the q-multinomial coefficient. Also see Carlitz [1963a], Kadell [1985a], and Knuth [1973, p. 33, Ex. 16]. Gasper derived the q-multinomial theorem in part (ii) several years ago by using the q-binomial theorem and mathematical induction. Andrews observed in a 1988 letter that it can also be derived by using the expansion formula for the q-Lauricella function Φ_D stated in Andrews [1972, (4.1)] and the q-Vandermonde sum. Some sums of q-multinomial coefficients are considered in Bressoud [1978, 1981c]. See also Agarwal [1953a].

Ex. 1.8 Jain [1980c] showed that the sum in this exercise is equivalent to the sum of a certain ${}_2\psi_2$ series, and summed some other ${}_2\psi_2$ series.

Ex. 1.10 Analogous recurrence relations for ${}_1\phi_1$ series are given in Slater [1954c].

Exercises 1.12 and 1.13 The notations Δ_q, ϑ_q, and D_q are also employed in the literature for this q-derivative operator. We employed the script $\mathcal{D}_q$ operator notation to distinguish this q-derivative operator from the q-derivative operator defined in (7.7.3) and the q-difference operator defined in Ex. 1.31. Additional results involving q-derivatives and q-difference equations are contained in Adams [1931], Agarwal [1953d], Andrews [1968, 1971a], Bowman [2002], Carmichael [1912], Di Vizio [2002, 2003], Faddeev and Kashaev [2002], Faddeev, Kashaev and Volkov [2001], Hahn [1949a,c, 1950, 1952, 1953], Ismail, Merkes and Styer [1990], Jackson [1905c, 1909a, 1910b,d,e], Miller [1970], Mimachi [1989], Sauloy [2003], Starcher [1931], and Trjitzinsky [1933]. For fractional q-derivatives and q-integrals see Agarwal [1969b] and Al-Salam and Verma [1975a,b]. Some "q-Taylor series" are considered in Jackson [1909b,c] and Wallisser [1985]. A q-Taylor theorem based on the sequence $\{\phi_n(x)\}_{n=0}^{\infty}$ with $\phi_n(x) = (ae^{i\theta}, ae^{-i\theta}; q)_n$, $x = \cos\theta$, was obtained by Ismail and Stanton [2003a,b] along with some interesting applications.

Ex. 1.14 For q-tangent and q-secant numbers and some of their properties, see Andrews and Foata [1980] and Foata [1981]. A discussion of q-trigonometry is given in Gosper [2001]. See also Bustoz and Suslov [1998] and Suslov [2003].

Exercises 1.20–1.23 Ismail and Muldoon [1994] studied some inequalities and monotonicity properties of the gamma and q-gamma functions.

Ex. 1.22 Also see Artin [1964, pp. 14–15]. A different characterization of Γ_q is presented in Kairies and Muldoon [1982].

Exercises 1.24–1.27 Other formulas involving q-Bessel functions are contained in Jackson [1904a–d, 1908], Ismail and Muldoon [1988], Rahman [1987, 1988c, 1989b,c], and Swarttouw and Meijer [1994]. It was pointed out by Ismail in an unpublished preprint in 1999 (rewritten for publication as Ismail [2003c]) that $J_\nu^{(3)}(x; q)$ was actually introduced by Jackson [1905a], contrary to the claim in Swarttouw [1992] that a special case of it was first discovered by Hahn [1953] and then in full generality by Exton [1978].

Ex. 1.28 See the generating functions for the continuous q-Hermite polynomials derived in Carlitz [1963b, 1972] and Bressoud [1980b], and the applications to modular forms in Bressoud [1986]. An extension of these q-Bessel functions to a q-quadratic grid is given in Ismail, Masson and Suslov [1999].

Ex. 1.32 Masuda *et al.* [1991] showed that the matrix elements that arise in the representations of certain quantum groups are expressible in terms of little q-Jacobi polynomials, and that this and a form of the Peter-Weyl theorem imply the orthogonality relation for these polynomials. Padé approximants for the moment generating function for the little q-Jacobi polynomials are employed in Andrews, Goulden and D.M. Jackson [1986] to explain and extend Shank's method for accelerating the convergence of sequences. Padé approximations for some q-hypergeometric functions are considered in Ismail, Perline and Wimp [1992].