7

Hankel Transforms and Their Applications

> "In most sciences one generation tears down what another has built, and what one has established, another undoes. In mathematics alone each generation adds a new storey to the old structure."
>
> Hermann Hankel

> "I have always regarded mathematics as an object of amusement rather than of ambition, and I can assure you that I enjoy the works of others much more than my own."
>
> Joseph-Louis Lagrange

7.1 Introduction

Hermann Hankel (1839–1873), a German mathematician, is remembered for his numerous contributions to mathematical analysis including the Hankel transformation, which occurs in the study of functions which depend only on the distance from the origin. He also studied functions, now named Hankel functions or Bessel functions of the third kind. The Hankel transform involving Bessel functions as the kernel arises naturally in axisymmetric problems formulated in cylindrical polar coordinates. This chapter deals with the definition and basic operational properties of the Hankel transform. A large number of axisymmetric problems in cylindrical polar coordinates are solved with the aid of the Hankel transform. The use of the joint Laplace and Hankel transforms is illustrated by several examples of applications to partial differential equations.

7.2 The Hankel Transform and Examples

We introduce the definition of the *Hankel transform* from the two-dimensional Fourier transform and its inverse given by

$$\mathscr{F}\{f(x,y)\} = F(k,l) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\{-i(\boldsymbol{\kappa}\cdot\mathbf{r})\}f(x,y)\,dx\,dy, \quad (7.2.1)$$

$$\mathscr{F}^{-1}\{F(k,l)\} = f(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\{i(\boldsymbol{\kappa}\cdot\mathbf{r})\}F(k,l)\,dk\,dl, \quad (7.2.2)$$

where $\mathbf{r} = (x,y)$ and $\boldsymbol{\kappa} = (k,l)$. Introducing polar coordinates $(x,y)=r(\cos\theta, \sin\theta)$ and $(k,l) = \kappa(\cos\phi, \sin\phi)$, we find $\boldsymbol{\kappa}\cdot\mathbf{r} = \kappa r\cos(\theta-\phi)$ and then

$$F(\kappa,\phi) = \frac{1}{2\pi}\int_0^{\infty} r\,dr \int_0^{2\pi} \exp[-i\kappa\, r\,\cos(\theta-\phi)]f(r,\theta)d\theta. \quad (7.2.3)$$

We next assume $f(r,\theta) = \exp(in\theta)f(r)$, which is not a very severe restriction, and make a change of variable $\theta - \phi = \alpha - \frac{\pi}{2}$ to reduce (7.2.3) to the form

$$F(\kappa,\phi) = \frac{1}{2\pi}\int_0^{\infty} rf(r)dr \times \int_{\phi_0}^{2\pi+\phi_0} \exp\left[in\left(\phi - \frac{\pi}{2}\right) + i(n\alpha - \kappa r\sin\alpha)\right]d\alpha, \quad (7.2.4)$$

where $\phi_0 = \left(\frac{\pi}{2} - \phi\right)$.

Using the integral representation of the Bessel function of order n

$$J_n(\kappa r) = \frac{1}{2\pi}\int_{\phi_0}^{2\pi+\phi_0} \exp[i(n\alpha - \kappa r\sin\alpha)]d\alpha \quad (7.2.5)$$

integral (7.2.4) becomes

$$F(\kappa,\phi) = \exp\left[in\left(\phi - \frac{\pi}{2}\right)\right]\int_0^{\infty} rJ_n(\kappa r)f(r)dr \quad (7.2.6)$$

$$= \exp\left[in\left(\phi - \frac{\pi}{2}\right)\right]\tilde{f}_n(\kappa), \quad (7.2.7)$$

where $\tilde{f}_n(\kappa)$ is called the *Hankel transform* of $f(r)$ and is defined formally by

$$\mathscr{H}_n\{f(r)\} = \tilde{f}_n(\kappa) = \int_0^\infty rJ_n(\kappa r)f(r)\,dr. \tag{7.2.8}$$

Similarly, in terms of the polar variables with the assumption $f(x,y) = f(r,\theta) = e^{in\theta}f(r)$ with (7.2.7), the inverse Fourier transform (7.2.2) becomes

$$\begin{aligned} e^{in\theta}f(r) &= \frac{1}{2\pi}\int_0^\infty \kappa\,d\kappa \int_0^{2\pi} \exp[i\kappa r\cos(\theta-\phi)]F(\kappa,\phi)d\phi \\ &= \frac{1}{2\pi}\int_0^\infty \kappa\,\tilde{f}_n(\kappa)d\kappa \int_0^{2\pi} \exp\left[in\left(\phi-\frac{\pi}{2}\right)+i\kappa r\cos(\theta-\phi)\right]d\phi, \end{aligned}$$

which is, by the change of variables $\theta-\phi = -\left(\alpha+\frac{\pi}{2}\right)$ and $\theta_0 = -\left(\theta+\frac{\pi}{2}\right)$,

$$\begin{aligned} &= \frac{1}{2\pi}\int_0^\infty \kappa\tilde{f}_n(\kappa)d\kappa \int_{\theta_0}^{2\pi+\theta_0} \exp[in(\theta+\alpha)-i\kappa r\sin\alpha]d\alpha \\ &= e^{in\theta}\int_0^\infty \kappa J_n(\kappa r)\tilde{f}_n(\kappa)d\kappa, \qquad \text{by (7.2.5).} \end{aligned} \tag{7.2.9}$$

Thus, the *inverse Hankel transform* is defined by

$$\mathscr{H}_n^{-1}\left[\tilde{f}_n(\kappa)\right] = f(r) = \int_0^\infty \kappa J_n(\kappa r)\tilde{f}_n(\kappa)d\kappa. \tag{7.2.10}$$

Instead of $\tilde{f}_n(\kappa)$, we often simply write $\tilde{f}(\kappa)$ for the Hankel transform specifying the order. Integrals (7.2.8) and (7.2.10) exist for certain large classes of functions, which usually occur in physical applications.

Alternatively, the famous Hankel integral formula (Watson, 1944, p. 453)

$$f(r) = \int_0^\infty \kappa\,J_n(\kappa r)d\kappa \int_0^\infty pJ_n(\kappa p)f(p)dp, \tag{7.2.11}$$

can be used to define the Hankel transform (7.2.8) and its inverse (7.2.10).

In particular, the Hankel transforms of the zero order ($n=0$) and of order one ($n=1$) are often useful for the solution of problems involving Laplace's equation in an axisymmetric cylindrical geometry.

Example 7.2.1 Obtain the zero-order Hankel transforms of

(a) $r^{-1}\exp(-ar)$, (b) $\dfrac{\delta(r)}{r}$, (c) $H(a-r)$,

where $H(r)$ is the Heaviside unit step function.

We have

(a) $\tilde{f}(\kappa) = \mathscr{H}_0\left\{\dfrac{1}{r}\exp(-ar)\right\} = \displaystyle\int_0^\infty \exp(-ar)\, J_0(\kappa r)dr = \frac{1}{\sqrt{\kappa^2+a^2}}.$

(b) $\tilde{f}(\kappa) = \mathscr{H}_0\left\{\dfrac{\delta(r)}{r}\right\} = \displaystyle\int_0^\infty \delta(r)J_0(\kappa r)dr = 1.$

(c) $$\tilde{f}(\kappa) = \mathscr{H}_0\{H(a-r)\} = \int_0^a rJ_0(\kappa r)dr = \frac{1}{\kappa^2}\int_0^{a\kappa} pJ_0(p)dp$$
$$= \frac{1}{\kappa^2}[pJ_1(p)]_0^{a\kappa} = \frac{a}{\kappa}J_1(a\kappa).$$

$\square$

Example 7.2.2 Find the first-order Hankel transforms of

(a) $f(r) = e^{-ar}$, (b) $f(r) = \dfrac{1}{r}\,e^{-ar}$, (c) $f(r) = \dfrac{\sin ar}{r}$.

We can write

(a) $\tilde{f}(\kappa) = \mathscr{H}_1\{e^{-ar}\} = \displaystyle\int_0^\infty re^{-ar}J_1(\kappa r)dr = \frac{\kappa}{(a^2+\kappa^2)^{\frac{3}{2}}}.$

(b) $\tilde{f}(\kappa) = \mathscr{H}_1\left\{\dfrac{a^{-ar}}{r}\right\} = \displaystyle\int_0^\infty e^{-ar}J_1(\kappa r)dr = \frac{1}{\kappa}\left[1 - a(\kappa^2+a^2)^{-\frac{1}{2}}\right].$

(c) $\tilde{f}(\kappa) = \mathscr{H}_1\left\{\dfrac{\sin ar}{r}\right\} = \displaystyle\int_0^\infty \sin ar\, J_1(\kappa r)\, dr = \frac{a\,H(\kappa-a)}{\kappa(\kappa^2-a^2)^{\frac{1}{2}}}.$

$\square$

Example 7.2.3 Find the nth $(n > -1)$ order Hankel transforms of

(a) $f(r) = r^n H(a-r)$, (b) $f(r) = r^n\exp(-ar^2)$.

Here we have, for $n > -1$,

$$\text{(a) } \tilde{\mathrm{f}}(\kappa) = \mathscr{H}_n\left[r^n H(a-r)\right] = \int_0^a r^{n+1} J_n(\kappa r)dr = \frac{a^{n+1}}{\kappa} J_{n+1}(a\kappa).$$

$$\text{(b) } \tilde{\mathrm{f}}(\kappa) = \mathscr{H}_n\left[r^n \exp(-ar^2)\right] = \int_0^\infty r^{n+1} J_n(\kappa r)\exp(-ar^2)dr$$

$$= \frac{\kappa^n}{(2a)^{n+1}} \exp\left(-\frac{\kappa^2}{4a}\right).$$

∎

7.3 Operational Properties of the Hankel Transform

THEOREM 7.3.1 *(Scaling)***.**
If $\mathscr{H}_n\{f(r)\} = \tilde{f}_n(\kappa)$, then

$$\mathscr{H}_n\{f(ar)\} = \frac{1}{a^2}\tilde{f}_n\left(\frac{\kappa}{a}\right), \qquad a > 0. \tag{7.3.1}$$

PROOF We have, by definition,

$$\mathscr{H}_n\{f(ar)\} = \int_0^\infty rJ_n(\kappa r)f(ar)dr$$

$$= \frac{1}{a^2}\int_0^\infty sJ_n\left(\frac{\kappa}{a}s\right)f(s)ds = \frac{1}{a^2}\tilde{f}_n\left(\frac{\kappa}{a}\right).$$

∎

THEOREM 7.3.2 *(Parseval's Relation)***.**
If $\tilde{f}(\kappa) = \mathscr{H}_n\{f(r)\}$ and $\tilde{g}(\kappa) = \mathscr{H}_n\{g(r)\}$, then

$$\int_0^\infty rf(r)g(r)dr = \int_0^\infty \kappa\tilde{f}(\kappa)\tilde{g}(\kappa)d\kappa. \tag{7.3.2}$$

PROOF We proceed formally to obtain

$$\int_0^\infty \kappa \tilde{f}(\kappa)\tilde{g}(\kappa)d\kappa = \int_0^\infty \kappa \tilde{f}(\kappa)d\kappa \int_0^\infty rJ_n(\kappa r)g(r)dr,$$

which is, interchanging the order of integration,

$$= \int_0^\infty rg(r)dr \int_0^\infty \kappa J_n(\kappa r)\tilde{f}(\kappa)d\kappa$$

$$= \int_0^\infty rg(r)f(r)dr.$$

∎

THEOREM 7.3.3 *(Hankel Transforms of Derivatives).*
If $\tilde{f}_n(\kappa) = \mathscr{H}_n\{f(r)\}$, then

$$\mathscr{H}_n\{f'(r)\} = \frac{\kappa}{2n}\left[(n-1)\tilde{f}_{n+1}(\kappa) - (n+1)\tilde{f}_{n-1}(\kappa)\right], \quad n \geq 1, \tag{7.3.3}$$

$$\mathscr{H}_1\{f'(r)\} = -\kappa \tilde{f}_0(\kappa), \tag{7.3.4}$$

provided $[rf(r)]$ vanishes as $r \to 0$ and $r \to \infty$.

PROOF We have, by definition,

$$\mathscr{H}_n\{f'(r)\} = \int_0^\infty rJ_n(\kappa r)f'(r)dr$$

which is, integrating by parts,

$$= [rf(r)J_n(\kappa r)]_0^\infty - \int_0^\infty f(r)\frac{d}{dr}[rJ_n(\kappa r)]dr. \tag{7.3.5}$$

We now use the properties of the Bessel function

$$\begin{aligned}\frac{d}{dr}[rJ_n(\kappa r)] &= J_n(\kappa r) + r\kappa J_n'(\kappa r) = J_n(\kappa r) + r\kappa J_{n-1}(\kappa r) - nJ_n(\kappa r)\\ &= (1-n)J_n(\kappa r) + r\kappa J_{n-1}(\kappa r).\end{aligned} \tag{7.3.6}$$

In view of the given condition, the first term of (7.3.5) vanishes as $r \to 0$ and $r \to \infty$, and the derivative within the integral in (7.3.5) can be replaced

by (7.3.6) so that (7.3.5) becomes

$$\mathscr{H}_n\{f'(r)\} = (n-1)\int_0^\infty f(r)J_n(\kappa r)dr - \kappa\tilde{f}_{n-1}(\kappa). \tag{7.3.7}$$

We next use the standard recurrence relation for the Bessel function

$$J_n(\kappa r) = \frac{\kappa r}{2n}[J_{n-1}(\kappa r) + J_{n+1}(\kappa r)]. \tag{7.3.8}$$

Thus, (7.3.7) can be rewritten as

$$\begin{aligned}\mathscr{H}_n[f'(r)] &= -\kappa\tilde{f}_{n-1}(\kappa) + \kappa\left(\frac{n-1}{2n}\right)\left[\int_0^\infty rf(r)\{J_{n-1}(\kappa r) + J_{n+1}(\kappa r)\}dr\right] \\ &= -\kappa\tilde{f}_{n-1}(\kappa) + \kappa\left(\frac{n-1}{2n}\right)\left[\tilde{f}_{n-1}(\kappa) + \tilde{f}_{n+1}(\kappa)\right] \\ &= \left(\frac{\kappa}{2n}\right)\left[(n-1)\tilde{f}_{n+1}(\kappa) - (n+1)\tilde{f}_{n-1}(\kappa)\right].\end{aligned}$$

In particular, when $n=1$, (7.3.4) follows immediately.

Similarly, repeated applications of (7.3.3) lead to the following result

$$\begin{aligned}\mathscr{H}_n\{f''(r)\} &= \frac{\kappa}{2n}\left[(n-1)\mathscr{H}_{n+1}\{f'(r)\} - (n+1)\mathscr{H}_{n-1}\{f'(r)\}\right] \\ &= \frac{\kappa^2}{4}\left[\left(\frac{n+1}{n-1}\right)\tilde{f}_{n-2}(\kappa) - 2\left(\frac{n^2-3}{n^2-1}\right)\tilde{f}_n(\kappa)\right. \\ &\qquad\left. + \left(\frac{n-1}{n+1}\right)\tilde{f}_{n+2}(\kappa)\right].\end{aligned} \tag{7.3.9}$$

∎

THEOREM 7.3.4 If $\mathscr{H}_n\{f(r)\} = \tilde{f}_n(\kappa)$**,** then

$$\mathscr{H}_n\left\{\left(\nabla^2 - \frac{n^2}{r^2}\right)f(r)\right\} = \mathscr{H}_n\left\{\frac{1}{r}\frac{d}{dr}\left(r\frac{df}{dr}\right) - \frac{n^2}{r^2}f(r)\right\} = -\kappa^2\tilde{f}_n(\kappa), \tag{7.3.10}$$

provided both $rf'(r)$ and $rf(r)$ vanish as $r \to 0$ and $r \to \infty$.

PROOF We have, by definition (7.2.8),

$$\begin{aligned}\mathscr{H}_n\left\{\frac{1}{r}\frac{d}{dr}\left(r\frac{df}{dr}\right) - \frac{n^2}{r^2}f(r)\right\} &= \int_0^\infty J_n(\kappa r)\left[\frac{d}{dr}\left(r\frac{df}{dr}\right)\right]dr \\ &\qquad - \int_0^\infty \frac{n^2}{r^2}[rJ_n(\kappa r)]f(r)dr,\end{aligned}$$

which is, invoking integration by parts,

$$= \left[\left(r\frac{df}{dr}\right) J_n(\kappa r)\right]_0^\infty - \kappa \int_0^\infty r\frac{df}{dr} J_n'(\kappa r)dr - \int_0^\infty \frac{n^2}{r^2}[rJ_n(\kappa r)]f(r)dr,$$

which is, by replacing the first term with zero because of the given assumption, and by invoking integration by parts again,

$$= -\left[\kappa r f(r) J_n'(\kappa r)\right]_0^\infty + \int_0^\infty \frac{d}{dr}\left[\kappa r J_n'(\kappa r)\right] f(r)dr - \int_0^\infty \frac{n^2}{r^2}[rJ_n(\kappa r)]f(r)dr.$$

We use the given assumptions and Bessel's differential equation,

$$\frac{d}{dr}\left[\kappa r\, J_n'(\kappa r)\right] + r\left(\kappa^2 - \frac{n^2}{r^2}\right) J_n(\kappa r) = 0, \tag{7.3.11}$$

to obtain

$$\begin{aligned}\mathscr{H}_n\left\{\left(\nabla^2 - \frac{n^2}{r^2}\right) f(r)\right\} &= -\int_0^\infty \left(\kappa^2 - \frac{n^2}{r^2}\right) r f(r) J_n(\kappa r)dr \\ &\quad - \int_0^\infty \frac{n^2}{r^2}[r f(r)]\, J_n(\kappa r)\, dr \\ &= -\kappa^2 \int_0^\infty r J_n(\kappa r) f(r)dr = -\kappa^2 \mathscr{H}_n[f(r)] = -\kappa^2 \tilde{f}_n(\kappa).\end{aligned}$$

This proves the theorem.

In particular, when $n = 0$ and $n = 1$, we obtain

$$\mathscr{H}_0\left\{\frac{1}{r}\frac{d}{dr}\left(r\frac{df}{dr}\right)\right\} = -\kappa^2 \tilde{f}_0(\kappa), \tag{7.3.12}$$

$$\mathscr{H}_1\left\{\frac{1}{r}\frac{d}{dr}\left(r\frac{df}{dr}\right) - \frac{1}{r^2} f(r)\right\} = -\kappa^2 \tilde{f}_1(\kappa). \tag{7.3.13}$$

Results (7.3.10), (7.3.12), and (7.3.13) are widely used for finding solutions of partial differential equations in axisymmetric cylindrical configurations. We illustrate this point by considering several examples of applications. ∎

7.4 Applications of Hankel Transforms to Partial Differential Equations

The Hankel transforms are extremely useful in solving a variety of partial differential equations in cylindrical polar coordinates. The following examples

illustrate applications of the Hankel transforms. The examples given here are only representative of a whole variety of physical problems that can be solved in a similar way.

Example 7.4.1 (*Free Vibration of a Large Circular Membrane*).
Obtain the solution of the free vibration of a large circular elastic membrane governed by the initial value problem

$$c^2\left(\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}\right)=\frac{\partial^2 u}{\partial t^2},\quad 0<r<\infty,\quad t>0, \tag{7.4.1}$$

$$u(r,0)=f(r),\quad u_t(r,0)=g(r),\quad \text{for } 0\le r<\infty, \tag{7.4.2ab}$$

where $c^2=(T/\rho)=$ constant, T is the tension in the membrane, and ρ is the surface density of the membrane.

Application of the zero-order Hankel transform with respect to r

$$\tilde{u}(\kappa,t)=\int_0^\infty r\,J_0(\kappa r)u(r,t)dr, \tag{7.4.3}$$

to (7.4.1)–(7.4.2ab) gives

$$\frac{d^2\tilde{u}}{dt^2}+c^2\kappa^2\tilde{u}=0, \tag{7.4.4}$$

$$\tilde{u}(\kappa,0)=\tilde{f}(\kappa),\qquad \tilde{u}_t(\kappa,0)=\tilde{g}(\kappa). \tag{7.4.5ab}$$

The general solution of this transformed system is

$$\tilde{u}(\kappa,t)=\tilde{f}(\kappa)\cos(c\kappa t)+(c\kappa)^{-1}\,\tilde{g}(\kappa)\sin(c\kappa t). \tag{7.4.6}$$

The inverse Hankel transform leads to the solution

$$u(r,t)=\int_0^\infty \kappa\,\tilde{f}(\kappa)\cos(c\kappa t)J_0(\kappa r)d\kappa$$
$$+\frac{1}{c}\int_0^\infty \tilde{g}(\kappa)\sin(c\kappa t)J_0(\kappa r)d\kappa. \tag{7.4.7}$$

In particular, we consider

$$u(r,0)=f(r)=Aa(r^2+a^2)^{-\frac{1}{2}},\quad u_t(r,0)=g(r)=0, \tag{7.4.8ab}$$

so that $\tilde{g}(\kappa)\equiv 0$ and

$$\tilde{f}(\kappa)=Aa\int_0^\infty r(a^2+r^2)^{-\frac{1}{2}}J_0(\kappa r)dr=\frac{Aa}{\kappa}e^{-a\kappa},\qquad \text{by Example 7.2.1(a).}$$

Thus, the formal solution (7.4.7) becomes

$$u(r,t) = Aa\int_0^\infty e^{-a\kappa}J_0(\kappa r)\cos(c\kappa t)d\kappa = Aa\,\mathrm{Re}\int_0^\infty \exp[-\kappa(a+ict)]J_0(\kappa r)d\kappa$$

$$= Aa\,\mathrm{Re}\left\{r^2+(a+ict)^2\right\}^{-\frac{1}{2}}, \qquad \text{by Example 7.2.1(a).} \tag{7.4.9}$$

□

Example 7.4.2 (*Steady Temperature Distribution in a Semi-Infinite Solid with a Steady Heat Source*).
Find the solution of the Laplace equation for the steady temperature distribution $u(r,z)$ with a steady and symmetric heat source $Q_0q(r)$:

$$u_{rr}+\frac{1}{r}u_r+u_{zz} = -Q_0q(r), \quad 0<r<\infty, \quad 0<z<\infty, \tag{7.4.10}$$

$$u(r,0) = 0, \quad 0<r<\infty, \tag{7.4.11}$$

where Q_0 is a constant. This boundary condition represents zero temperature at the boundary $z=0$.

Application of the zero-order Hankel transform to (7.4.10) and (7.4.11) gives

$$\frac{d^2\tilde{u}}{dz^2}-\kappa^2\tilde{u} = -Q_0\tilde{q}(\kappa), \quad \tilde{u}(\kappa,0)=0.$$

The bounded general solution of this system is

$$\tilde{u}(\kappa,z) = A\exp(-\kappa z)+\frac{Q_0}{\kappa^2}\tilde{q}(\kappa),$$

where A is a constant to be determined from the transformed boundary condition. In this case

$$A = -\frac{Q_0}{\kappa^2}\tilde{q}(\kappa).$$

Thus, the formal solution is

$$\tilde{u}(\kappa,z) = \frac{Q_0\,\tilde{q}(\kappa)}{\kappa^2}(1-e^{-\kappa z}). \tag{7.4.12}$$

The inverse Hankel transform yields the exact integral solution

$$u(r,z) = Q_0\int_0^\infty \frac{\tilde{q}(\kappa)}{\kappa}(1-e^{-\kappa z})J_0(\kappa r)d\kappa. \tag{7.4.13}$$

□

Example 7.4.3 (*Axisymmetric Diffusion Equation*).
Find the solution of the axisymmetric diffusion equation

$$u_t = \kappa\left(u_{rr} + \frac{1}{r}u_r\right), \qquad 0 < r < \infty, \quad t > 0, \tag{7.4.14}$$

where $\kappa(>0)$ is a diffusivity constant and

$$u(r,0) = f(r), \qquad \text{for} \quad 0 < r < \infty. \tag{7.4.15}$$

We apply the zero-order Hankel transform defined by (7.4.3) to obtain

$$\frac{d\tilde{u}}{dt} + k^2\kappa\tilde{u} = 0, \qquad \tilde{u}(k,0) = \tilde{f}(k),$$

where k is the Hankel transform variable. The solution of this transformed system is

$$\tilde{u}(k,t) = \tilde{f}(k)\exp(-\kappa k^2 t). \tag{7.4.16}$$

Application of the inverse Hankel transform gives

$$u(r,t) = \int_0^\infty k\tilde{f}(k)J_0(kr)e^{-\kappa k^2 t}dk = \int_0^\infty k\left[\int_0^\infty lJ_0(kl)f(l)dl\right]e^{-\kappa k^2 t}J_0(kr)dk$$

which is, interchanging the order of integration,

$$= \int_0^\infty l\,f(l)dl\int_0^\infty k\,J_0(kl)J_0(kr)\exp(-\kappa k^2 t)dk. \tag{7.4.17}$$

Using a standard table of integrals involving Bessel functions, we state

$$\int_0^\infty kJ_0(kl)\,J_0(kr)\exp(-k^2\kappa t)dk = \frac{1}{2\kappa t}\exp\left[-\frac{(r^2+l^2)}{4\kappa t}\right]I_0\left(\frac{rl}{2\kappa t}\right), \tag{7.4.18}$$

where $I_0(x)$ is the modified Bessel function and $I_0(0) = 1$. In particular, when $l = 0$, $J_0(0) = 1$ and integral (7.4.18) becomes

$$\int_0^\infty kJ_0(kr)\exp(-k^2\kappa t)dk = \frac{1}{2\kappa t}\exp\left(-\frac{r^2}{4\kappa t}\right). \tag{7.4.19}$$

We next use (7.4.18) to rewrite (7.4.17) as

$$u(r,t) = \frac{1}{2\kappa t}\int_0^\infty lf(l)I_0\left(\frac{rl}{2\kappa t}\right)\exp\left[-\frac{(r^2+l^2)}{4\kappa t}\right]dl. \tag{7.4.20}$$

We now assume $f(r)$ to represent a heat source concentrated in a circle of radius a and allow $a \to 0$ so that the heat source is concentrated at $r=0$ and

$$\lim_{a\to 0} 2\pi \int_0^a r f(r) dr = 1.$$

Or, equivalently,

$$f(r) = \frac{1}{2\pi}\frac{\delta(r)}{r},$$

where $\delta(r)$ is the Dirac delta function.

Thus, the final solution due to the concentrated heat source at $r=0$ is

$$\begin{aligned} u(r,t) &= \frac{1}{4\pi\kappa t}\int_0^\infty \delta(l) I_0\left(\frac{rl}{2\kappa t}\right)\exp\left[-\frac{r^2+l^2}{4\kappa t}\right] dl \\ &= \frac{1}{4\pi\kappa t}\exp\left(-\frac{r^2}{4\kappa t}\right). \end{aligned} \tag{7.4.21}$$

$\square$

Example 7.4.4 (*Axisymmetric Acoustic Radiation Problem*).
Obtain the solution of the wave equation

$$c^2\left(u_{rr} + \frac{1}{r}u_r + u_{zz}\right) = u_{tt}, \quad 0<r<\infty, \quad z>0, \quad t>0, \tag{7.4.22}$$

$$u_z = F(r,t) \quad \text{on } z=0, \tag{7.4.23}$$

where $F(r,t)$ is a given function and c is a constant. We also assume that the solution is bounded and behaves as outgoing spherical waves.

We seek a steady-state solution for the acoustic radiation potential $u = e^{i\omega t}\phi(r,z)$ with $F(r,t) = e^{i\omega t} f(r)$, so that ϕ satisfies the Helmholtz equation

$$\phi_{rr} + \frac{1}{r}\phi_r + \phi_{zz} + \left(\frac{\omega^2}{c^2}\right)\phi = 0, \quad 0<r<\infty, \quad z>0, \tag{7.4.24}$$

with the boundary condition

$$\phi_z = f(r) \qquad \text{on} \quad z=0, \tag{7.4.25}$$

where $f(r)$ is a given function of r.

Application of the Hankel transform $\mathscr{H}_0\{\phi(r,z)\} = \tilde{\phi}(k,z)$ to (7.4.24)–(7.4.25) gives

$$\tilde{\phi}_{zz} = \kappa^2\tilde{\phi}, \qquad z>0,$$

$$\tilde{\phi}_z = \tilde{f}(k), \qquad \text{on } z=0,$$

where

$$\kappa = \left(k^2 - \frac{\omega^2}{c^2}\right)^{\frac{1}{2}}.$$

The solution of this differential system is

$$\tilde{\phi}(k, z) = -\frac{1}{\kappa}\tilde{f}(k)\exp(-\kappa z), \tag{7.4.26}$$

where κ is real and positive for $k > \omega/c$, and purely imaginary for $k < \omega/c$.

The inverse Hankel transform yields the formal solution

$$\phi(r, z) = -\int_0^\infty \frac{k}{\kappa}\tilde{f}(k)\, J_0(kr)\exp(-\kappa z)dk. \tag{7.4.27}$$

Since the exact evaluation of this integral is difficult for an arbitrary $\tilde{f}(k)$, we choose a simple form of $f(r)$ as

$$f(r) = AH(a - r), \tag{7.4.28}$$

where A is a constant, and hence, $\tilde{f}(k) = \frac{Aa}{k}J_1(ak)$.

Thus, the solution (7.4.27) takes the form

$$\phi(r, z) = -Aa\int_0^\infty \frac{1}{\kappa}J_1(ak)J_0(kr)\exp(-\kappa z)dk. \tag{7.4.29}$$

For an asymptotic evaluation of this integral, it is convenient to express (7.4.29) in terms of R which is the distance from the z-axis so that $R^2 = r^2 + z^2$ and $z = R\cos\theta$. Using the asymptotic result for the Bessel function

$$J_0(kr) \sim \left(\frac{2}{\pi kr}\right)^{\frac{1}{2}}\cos\left(kr - \frac{\pi}{4}\right) \qquad \text{as} \quad r \to \infty, \tag{7.4.30}$$

where $r = R\sin\theta$. Consequently, (7.4.29) combined with $u = \exp(i\omega t)\phi$ becomes

$$u \sim -\frac{Aa\sqrt{2}e^{i\omega t}}{\sqrt{\pi R\sin\theta}}\int_0^\infty \frac{1}{\kappa\sqrt{k}}J_1(ak)\cos\left(kR\sin\theta - \frac{\pi}{4}\right)\exp(-\kappa z)dk.$$

This integral can be evaluated asymptotically for $R \to \infty$ using the stationary phase approximation formula to obtain the final result

$$u \sim -\frac{Aac}{\omega R\sin\theta}J_1(ak_1)\exp\left[i\left(\omega t - \frac{\omega R}{c}\right)\right], \tag{7.4.31}$$

where $k_1 = \omega/(c\sin\theta)$ is the stationary point. Physically, this solution represents outgoing spherical waves with constant velocity c and decaying amplitude as $R \to \infty$. ∎

Example 7.4.5 (*Axisymmetric Biharmonic Equation*).
We solve the axisymmetric boundary value problem

$$\nabla^4 u(r,z) = 0, \qquad 0 \le r < \infty, \quad z > 0\,, \tag{7.4.32}$$

with the boundary data

$$u(r,0) = f(r), \quad 0 \le r < \infty, \tag{7.4.33}$$

$$\frac{\partial u}{\partial z} = 0 \qquad \text{on } z = 0, \quad 0 \le r < \infty, \tag{7.4.34}$$

$$u(r,z) \to 0 \qquad \text{as } r \to \infty, \tag{7.4.35}$$

where the axisymmetric biharmonic operator is

$$\nabla^4 = \nabla^2(\nabla^2) = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}\right)\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}\right). \tag{7.4.36}$$

The use of the Hankel transform $\mathscr{H}_0\{u(r,z)\} = \tilde{u}(k,z)$ to this problem gives

$$\left(\frac{d^2}{dz^2} - k^2\right)^2 \tilde{u}(k,z) = 0, \quad z > 0, \tag{7.4.37}$$

$$\tilde{u}(k,0) = \tilde{f}(k), \quad \frac{d\tilde{u}}{dz} = 0 \qquad \text{on } z = 0. \tag{7.4.38}$$

The bounded solution of (7.4.37) is

$$\tilde{u}(k,z) = (A + zB)\exp(-kz), \tag{7.4.39}$$

where A and B are integrating constants to be determined by (7.4.38) as $A = \tilde{f}(k)$ and $B = k\tilde{f}(k)$. Thus, solution (7.4.39) becomes

$$\tilde{u}(k,z) = (1 + kz)\tilde{f}(k)\exp(-kz). \tag{7.4.40}$$

The inverse Hankel transform gives the formal solution

$$u(r,z) = \int_0^\infty k(1+kz)\tilde{f}(k)J_0(kr)\exp(-kz)dk. \tag{7.4.41}$$

∎

Example 7.4.6 (*The Axisymmetric Cauchy-Poisson Water Wave Problem*).
We consider the initial value problem for an inviscid water of finite depth h with a free horizontal surface at $z = 0$, and the z-axis positive upward. We

assume that the liquid has constant density ρ with no surface tension. The surface waves are generated in water, which is initially at rest for $t<0$ by the prescribed free surface elevation. In cylindrical polar coordinates (r,θ,z), the axisymmetric water wave equations for the velocity potential $\phi(r,z,t)$ and the free surface elevation $\eta(r,t)$ are

$$\nabla^2\phi=\phi_{rr}+\frac{1}{r}\phi_r+\phi_{zz}=0,\quad 0\le r<\infty,\quad -h\le z\le 0,\quad t>0, \tag{7.4.42}$$

$$\left.\begin{aligned}\phi_z-\eta_t&=0\\ \phi_t+g\eta&=0\end{aligned}\right\}\quad \text{on } z=0,\quad t>0, \tag{7.4.43ab}$$

$$\phi_z=0\qquad \text{on } z=-h,\quad t>0. \tag{7.4.44}$$

The initial conditions are

$$\phi(r,0,0)=0\quad\text{and}\quad \eta(r,0)=\eta_0(r),\qquad \text{for } 0\le r<\infty, \tag{7.4.45}$$

where g is the acceleration due to gravity and $\eta_0(r)$ is the given free surface elevation.

We apply the joint Laplace and the zero-order Hankel transform defined by

$$\tilde{\bar{\phi}}(k,z,s)=\int_0^\infty e^{-st}dt\int_0^\infty rJ_0(kr)\phi(r,z,t)dr, \tag{7.4.46}$$

to (7.4.42)–(7.4.44) so that these equations reduce to

$$\left(\frac{d^2}{dz^2}-k^2\right)\tilde{\bar{\phi}}=0,$$

$$\left.\begin{aligned}\frac{d\tilde{\bar{\phi}}}{dz}-s\,\tilde{\bar{\eta}}&=-\tilde{\eta}_0(k)\\ s\,\tilde{\bar{\phi}}+g\,\tilde{\bar{\eta}}&=0\end{aligned}\right\}\quad\text{on } z=0,$$

$$\tilde{\bar{\phi}}_z=0\qquad\qquad \text{on } z=-h,$$

where $\tilde{\eta}_0(k)$ is the Hankel transform of $\eta_0(r)$ of order zero.

The solutions of this system are

$$\tilde{\bar{\phi}}(k,z,s)=-\frac{g\,\tilde{\eta}_0(k)}{(s^2+\omega^2)}\frac{\cosh k(z+h)}{\cosh kh}, \tag{7.4.47}$$

$$\tilde{\bar{\eta}}(k,s)=\frac{s\,\tilde{\eta}_0(k)}{(s^2+\omega^2)}, \tag{7.4.48}$$

where

$$\omega^2=gk\,\tanh(kh), \tag{7.4.49}$$

is the famous *dispersion relation* between frequency ω and wavenumber k for water waves in a liquid of depth h. Physically, this dispersion relation describes the interaction between the inertial and gravitational forces.

Application of the inverse transforms gives the integral solutions

$$\phi(r,z,t) = -g\int_0^\infty kJ_0(kr)\tilde{\eta}_0(k)\left(\frac{\sin\omega t}{\omega}\right)\frac{\cosh k(z+h)}{\cosh kh}dk, \quad (7.4.50)$$

$$\eta(r,t) = \int_0^\infty kJ_0(kr)\tilde{\eta}_0(k)\cos\omega t\,dk. \quad (7.4.51)$$

These wave integrals represent exact solutions for ϕ and η at any r and t, but the physical features of the wave motions cannot be described by them. In general, the exact evaluation of the integrals is almost a formidable task. In order to resolve this difficulty, it is necessary and useful to resort to asymptotic methods. It will be sufficient for the determination of the basic features of the wave motions to evaluate (7.4.50) or (7.4.51) asymptotically for a large time and distance with (r/t) held fixed. We now replace $J_0(kr)$ by its asymptotic formula (7.4.30) for $kr\to\infty$, so that (7.4.51) gives

$$\eta(r,t) \sim \left(\frac{2}{\pi r}\right)^{\frac{1}{2}}\int_0^\infty \sqrt{k}\,\tilde{\eta}_0(k)\cos\left(kr-\frac{\pi}{4}\right)\cos\omega t\,dk$$

$$= (2\pi r)^{-\frac{1}{2}}\,\mathrm{Re}\int_0^\infty \sqrt{k}\,\tilde{\eta}_0(k)\exp\left[i\left(\omega t - kr + \frac{\pi}{4}\right)\right]dk. \quad (7.4.52)$$

Application of the stationary phase method to (7.4.52) yields the solution

$$\eta(r,t) \sim \left[\frac{k_1}{rt|\omega''(k_1)|}\right]^{\frac{1}{2}}\tilde{\eta}_0(k_1)\cos[t\,\omega(k_1)-k_1 r], \quad (7.4.53)$$

where the stationary point $k_1 = \left(gt^2/4r^2\right)$ is the root of the equation

$$\omega'(k) = \frac{r}{t}. \quad (7.4.54)$$

For sufficiently deep water, $kh\to\infty$, the dispersion relation becomes

$$\omega^2 = gk. \quad (7.4.55)$$

The solution of the axisymmetric Cauchy-Poisson problem is based on a prescribed initial displacement of unit volume that is concentrated at the origin, which means that $\eta_0(r) = (a/2\pi r)\delta(r)$ so that $\tilde{\eta}_0(k) = \dfrac{a}{2\pi}$. Thus, the asymptotic solution is obtained from (7.4.53) in the form

$$\eta(r,t) \sim \frac{agt^2}{4\pi\sqrt{2}\,r^3}\cos\left(\frac{gt^2}{4r}\right), \quad gt^2 >> 4r. \quad (7.4.56)$$

It is noted that solution (7.4.53) is no longer valid when $\omega''(k_1) = 0$. This case can be handled by a modification of the asymptotic evaluation (see Debnath, 1994, p. 91).

A wide variety of other physical problems solved by the Hankel transform, and/or by the joint Hankel and Laplace transform are given in books by Sneddon (1951, 1972) and by Debnath (1994), and in research papers by Debnath (1969, 1983, 1989), Mohanti (1979), and Debnath and Rollins (1992) listed in the Bibliography. ∎

7.5 Exercises

1. Show that

   (a) $\mathscr{H}_0\{(a^2 - r^2)H(a - r)\} = \dfrac{4a}{\kappa^3} J_1(\kappa a) - \dfrac{2a^2}{\kappa^2} J_0(a\kappa)$,

   (b) $\mathscr{H}_n\{r^n e^{-ar}\} = \dfrac{a}{\sqrt{\pi}} \cdot 2^{n+1}\Gamma\left(n + \dfrac{3}{2}\right)\kappa^n (a^2 + \kappa^2)^{-\left(n+\frac{3}{2}\right)}$,

   (c) $\mathscr{H}_n\left\{\dfrac{2n}{r} f(r)\right\} = k\,\mathscr{H}_{n-1}\{f(r)\} + k\,\mathscr{H}_{n+1}\{f(r)\}$.

2. (a) Show that the solution of the boundary value problem

$$u_{rr} + \frac{1}{r} u_r + u_{zz} = 0, \quad 0 < r < \infty, \quad 0 < z < \infty,$$
$$u(r, z) = \frac{1}{\sqrt{a^2 + r^2}} \quad \text{on } z = 0, \quad 0 < r < \infty,$$

   is

$$u(r, z) = \int_0^\infty e^{-\kappa(z+a)} J_0(\kappa r) d\kappa = \frac{1}{\sqrt{(z + a)^2 + r^2}}.$$

   (b) Obtain the solution of the equation in $2(a)$ with $u(r, 0) = f(r) = H(a - r), \quad 0 < r < \infty$.

3. (a) The axisymmetric initial value problem is governed by

$$u_t = \kappa\left(u_{rr} + \frac{1}{r} u_r\right) + \delta(t) f(r), \ 0 < r < \infty, \ t > 0,$$
$$u(r, 0) = 0 \quad \text{for} \quad 0 < r < \infty.$$

   Show that the formal solution of this problem is

$$u(r, t) = \int_0^\infty k J_0(kr) \tilde{f}(k) \exp(-k^2 \kappa t) dk.$$

(b) For the special case when $f(r) = \left(\dfrac{Q}{\pi a^2}\right) H(a-r)$, show that the solution is

$$u(r,t) = \left(\frac{Q}{\pi a}\right) \int_0^\infty J_0(kr) J_1(ak) \exp(-k^2 \kappa t) dk.$$

4. If $f(r) = A(a^2 + r^2)^{-\frac{1}{2}}$ where A is a constant, show that the solution of the biharmonic equation described in Example 7.4.5 is

$$u(r,z) = A \frac{\{r^2 + (z+a)(2z+a)\}}{[r^2 + (z+a)^2]^{3/2}}.$$

5. Show that the solution of the boundary value problem

$$u_{rr} + \frac{1}{r} u_r + u_{zz} = 0, \quad 0 \le r < \infty, \quad z > 0,$$
$$u(r,0) = u_0 \qquad \text{for } 0 \le r \le a, \qquad u_0 \text{ is a constant,}$$
$$u(r,z) \to 0 \qquad \text{as } z \to \infty,$$

is

$$u(r,z) = a\, u_0 \int_0^\infty J_1(ak) J_0(kr) \exp(-kz) dk.$$

Find the solution of the problem when u_0 is replaced by an arbitrary function $f(r)$, and a by infinity.

6. Solve the axisymmetric biharmonic equation for the small-amplitude free vibration of a thin elastic disk

$$b^2 \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right)^2 u + u_{tt} = 0, \quad 0 < r < \infty, \quad t > 0,$$
$$u(r,0) = f(r), \quad u_t(r,0) = 0 \qquad \text{for } 0 < r < \infty,$$

where $b^2 = \left(\dfrac{D}{2\sigma h}\right)$ is the ratio of the flexural rigidity of the disk and its mass $2h\sigma$ per unit area.

7. Show that the zero-order Hankel transform solution of the axisymmetric Laplace equation

$$u_{rr} + \frac{1}{r} u_r + u_{zz} = 0, \quad 0 < r < \infty, \quad -\infty < z < \infty,$$

with the boundary data

$$\lim_{r \to 0} (r^2 u) = 0, \quad \lim_{t \to 0} (2\pi r) u_r = -f(z), \quad -\infty < z < \infty,$$

is

$$\tilde{u}(k,z) = \frac{1}{4\pi k} \int_{-\infty}^{\infty} \exp\{-k|z-\zeta|\} f(\zeta) d\zeta.$$

Hence, show that

$$u(r,z) = \frac{1}{4\pi} \int_{-\infty}^{\infty} \left\{r^2 + (z-\zeta)^2\right\}^{-\frac{1}{2}} f(\zeta) d\zeta.$$

8. Solve the nonhomogeneous diffusion problem

$$\begin{aligned} &u_t = \kappa \left(u_{rr} + \frac{1}{r} u_r \right) + Q(r,t), \quad 0 < r < \infty, \quad t > 0, \\ &u(r,0) = f(r) \qquad\qquad \text{for } 0 < r < \infty, \end{aligned}$$

where κ is a constant.

9. Solve the problem of the electrified unit disk in the $x-y$ plane with center at the origin. The electric potential $u(r,z)$ is axisymmetric and satisfies the boundary value problem

$$\begin{aligned} &u_{rr} + \frac{1}{r} u_r + u_{zz} = 0, \quad 0 < r < \infty, \quad 0 < z < \infty, \\ &u(r,0) = u_0, \quad 0 \le r < a, \\ &\frac{\partial u}{\partial z} = 0, \quad \text{on } z = 0 \quad \text{for } a < r < \infty, \\ &u(r,z) \to 0 \quad \text{as } z \to \infty \text{ for all } r, \end{aligned}$$

where u_0 is constant. Show that the solution is

$$u(r,z) = \left(\frac{2au_0}{\pi} \right) \int_0^{\infty} J_0(kr) \left(\frac{\sin ak}{k} \right) e^{-kz} dk.$$

10. Solve the axisymmetric surface wave problem in deep water due to an oscillatory surface pressure. The governing equations are

$$\nabla^2 \phi = \phi_{rr} + \frac{1}{r} \phi_r + \phi_{zz} = 0, \quad 0 \le r < \infty, \quad -\infty < z \le 0,$$

$$\left.\begin{aligned} &\phi_t + g\eta = -\frac{P}{\rho} p(r) \exp(i\omega t) \\ &\phi_z - \eta_t = 0 \end{aligned}\right\} \text{on } z = 0, \quad t > 0,$$

$$\phi(r,z,0) = 0 = \eta(r,0), \quad \text{for } 0 \le r < \infty, \quad -\infty < z \le 0.$$

11. Solve the Neumann problem for the Laplace equation

$$u_{rr} + \frac{1}{r}\, u_r + u_{zz} = 0, \quad 0 < r < \infty, \quad 0 < z < \infty$$
$$u_z(r, 0) = -\frac{1}{\pi a^2} H(a - r), \quad 0 < r < \infty$$
$$u(r, z) \to 0 \quad \text{as } z \to \infty \quad \text{for } 0 < r < \infty.$$

Show that

$$\lim_{a \to 0} u(r, z) = \frac{1}{2\pi} (r^2 + z^2)^{-\frac{1}{2}}.$$

12. Solve the Cauchy problem for the wave equation in a dissipating medium

$$u_{tt} + 2\kappa u_t = c^2 \left(u_{rr} + \frac{1}{r}\, u_r \right), \quad 0 < r < \infty, \quad t > 0,$$
$$u(r, 0) = f(r), \quad u_t(r, 0) = g(r) \quad \text{for } 0 < r < \infty,$$

where κ is a constant.

13. Use the joint Laplace and Hankel transform to solve the initial-boundary value problem

$$c^2 \left(u_{rr} + \frac{1}{r} u_r + u_{zz} \right) = u_{tt}, \quad 0 < r < \infty, \quad 0 < z < \infty, \quad t > 0,$$
$$u_z(r, 0, t) = H(a - r) H(t), \quad 0 < r < \infty, \quad t > 0,$$
$$u(r, z, t) \to 0 \quad \text{as } r \to \infty \text{ and } u(r, z, t) \to 0 \quad \text{as } z \to \infty,$$
$$u(r, z, 0) = 0 = u_t(r, z, 0),$$

and show that

$$u_t(r, z, t) = -ac\, H\left(t - \frac{z}{c}\right) \int_0^\infty J_1(ak)\, J_0 \left\{ ck \sqrt{t^2 - \frac{z^2}{c^2}} \right\} J_0(kr) dk.$$

14. Find the steady temperature $u(r, z)$ in a beam $0 \le r < \infty$, $0 \le z \le a$ when the face $z = 0$ is kept at temperature $u(r, 0) = 0$, and the face $z = a$ is insulated except that heat is supplied through a circular hole such that

$$u_z(r, a) = H(b - r).$$

The temperature $u(r, z)$ satisfies the axisymmetric equation

$$u_{rr} + \frac{1}{r}\, u_r + u_{zz} = 0, \quad 0 \le r < \infty, \quad 0 \le z \le a.$$

15. Find the integral solution of the initial-boundary value problem

$$u_{rr} + \frac{1}{r}\, u_r + u_{zz} = u_t, \quad 0 \le r < \infty, \quad 0 \le z < \infty, \quad t > 0,$$
$$u(r, z, 0) = 0 \qquad \text{for all } r \text{ and } z,$$
$$\left(\frac{\partial u}{\partial r}\right)_{r=0} = 0, \qquad \text{for} \quad 0 \le z < \infty, \quad t > 0,$$
$$\left(\frac{\partial u}{\partial z}\right)_{z=0} = -\frac{H(a-r)}{\sqrt{a^2 + r^2}}, \qquad \text{for} \quad 0 < r < \infty, \quad 0 < t < \infty,$$
$$u(r, z, t) \to 0 \quad \text{as } r \to \infty \quad \text{or} \quad z \to \infty.$$

16. Heat is supplied at a constant rate Q per unit area per unit time over a circular area of radius a in the plane $z = 0$ to an infinite solid of thermal conductivity K, the rest of the plane is kept at zero temperature. Solve for the steady temperature field $u(r, z)$ that satisfies the Laplace equation

$$u_{rr} + \frac{1}{r}\, u_r + u_{zz} = 0, \quad 0 < r < \infty, \quad -\infty < z < \infty,$$

with the boundary conditions

$$u \to 0 \text{ as } r \to \infty, \quad u \to 0 \quad \text{as } |z| \to \infty,$$
$$-K u_z = \left(\frac{2Q}{\pi\, a^2}\right) H(a - r) \text{ when } z = 0.$$

17. The velocity potential $\phi(r, z)$ for the flow of an inviscid fluid through a circular aperture of unit radius in a plane rigid screen satisfies the Laplace equation

$$\phi_{rr} + \frac{1}{r}\phi_r + \phi_{zz} = 0, \; 0 < r < \infty$$

with the boundary conditions

$$\left.\begin{array}{ll} \phi = 1 & \text{for} \quad 0 < r < 1 \\ \phi_z = 0 & \text{for} \quad r > 1 \end{array}\right\} \text{ on } z = 0.$$

Obtain the solution of this boundary value problem.

18. Solve the Cauchy-Poisson wave problem (Debnath, 1989) for a viscous liquid of finite or infinite depth governed by the equations, free surface, boundary, and initial conditions

$$\phi_{rr} + \frac{1}{r}\phi_r + \phi_{zz} = 0,$$
$$\psi_t = \nu \left(\psi_{rr} + \frac{1}{r}\psi_r - \frac{1}{r^2}\psi + \psi_{zz}\right),$$

where $\phi(r,z,t)$ and $\psi(r,z,t)$ represent the potential and stream functions, respectively, $0 \le r < \infty$, $-h \le z \le 0$ (or $-\infty < z \le 0$) and $t > 0$.

The free surface conditions are

$$\left.\begin{aligned} &\eta_t - w = 0 \\ &\mu(u_z + w_r) = 0 \\ &\phi_t + g\eta + 2\nu\, w_z = 0 \end{aligned}\right\} \quad \text{on} \quad z = 0,\ t > 0$$

where $\eta = \eta(r,t)$ is the free surface elevation, $u = \phi_r + \psi_z$ and $w = \phi_z - \dfrac{\psi}{r} - \psi_r$ are the radial and vertical velocity components of liquid particles, $\mu = \rho\nu$ is the dynamic viscosity, ρ is the density, and ν is the kinematic viscosity of the liquid.

The boundary conditions at the rigid bottom are

$$\left.\begin{aligned} &u = \phi_r + \psi_z = 0 \\ &w = \phi_z - \frac{1}{r}(r\psi)_r = 0 \end{aligned}\right\} \text{ on } z = -h.$$

The initial conditions are

$$\eta = a\,\frac{\delta(r)}{r}, \quad \phi = \psi = 0 \quad \text{at } t = 0,$$

where a is a constant and $\delta(r)$ is the Dirac delta function.

If the liquid is of infinite depth, the bottom boundary conditions are

$$(\phi, \psi) \to (0,0) \qquad \text{as } z \to -\infty.$$

19. Use the joint Hankel and Laplace transform method to solve the initial-boundary value problem

$$u_{rr} + \frac{1}{r}\,u_r - u_{tt} - 2\varepsilon\, u_t = a\,\frac{\delta(r)}{r}\,\delta(t), \quad 0 < r < \infty, \quad t > 0,$$

$$u(r,t) \to 0 \qquad \text{as } r \to \infty,$$

$$u(0,t) \text{ is finite for } t > 0,$$

$$u(r,0) = 0 = u_t(r,0) \qquad \text{for } 0 < r < \infty.$$

20. Surface waves are generated in an inviscid liquid of infinite depth due to an explosion (Sen, 1963) above it, which generates the pressure field $p(r,t)$. The velocity potential $u = \phi(r,z,t)$ satisfies the Laplace equation

$$u_{rr} + \frac{1}{r}\,u_r + u_{zz} = 0, \quad 0 < r < \infty, \quad t > 0,$$

and the free surface condition

$$u_{tt} + g\, u_z = \frac{1}{\rho}\left(\frac{\partial p}{\partial t}\right)[H(r) - H\{r, r_0(t)\}] \qquad \text{on } z = 0,$$

where ρ is the constant density of the liquid, $r_0(t)$ is the extent of the blast, and the liquid is initially at rest.

Solve this problem.

21. The electrostatic potential $u(r, z)$ generated in the space between two horizontal disks at $z = \pm a$ by a point charge q at $r = z = 0$ is described by a singular function at $r = z = 0$ is

$$u(r, z) = \phi(r, z) + q(r^2 + z^2)^{-\frac{1}{2}},$$

where $\phi(r, z)$ satisfies the Laplace equation

$$\phi_{rr} + \frac{1}{r}\,\phi_r + \phi_{zz} = 0, \quad 0 < r < \infty$$

with the boundary conditions

$$\phi(r, z) = -q(r^2 + z^2)^{-\frac{1}{2}} \text{ at } z = \pm a.$$

Obtain the solution for $\phi(r, z)$ and then $u(r, z)$.

22. Show that

(a) $\mathscr{H}_n\left[e^{-ar} f(r)\right] = \mathscr{L}\left\{r f(r) J_n(\kappa r)\right\}$,

(b) $\mathscr{H}_0\left[e^{-ar^2} J_0(br)\right] = \dfrac{a}{2}\exp\left(\dfrac{\kappa^2 - b^2}{4a}\right) I_0\left(\dfrac{b\kappa}{2a}\right)$,

(c) $\mathscr{H}_n\left[r^{n-1} e^{-ar}\right] = \dfrac{(2\kappa)^n (n - \frac{1}{2})!}{\sqrt{\pi}(\kappa^2 + a^2)^{n+\frac{1}{2}}}$,

(d) $\mathscr{H}_n\left[\dfrac{f(r)}{r}\right] = \left(\dfrac{\kappa}{2n}\right)\left[\tilde{f}_{n-1}(\kappa) + \tilde{f}_{n+1}(\kappa)\right]$,

(e) $\mathscr{H}_n\left[r^{n-1}\dfrac{d}{dr}\left\{r^{1-n} f(r)\right\}\right] = -\kappa \tilde{f}_{n-1}(\kappa)$,

(f) $\mathscr{H}_n\left[r^{-(n+1)}\dfrac{d}{dr}\left\{r^{n+1} f(r)\right\}\right] = \kappa \tilde{f}_{n+1}(\kappa)$.

23. Show that

(a) $\mathscr{H}_0\left[e^{-\frac{r^2}{2}}\right] = e^{-\frac{\kappa^2}{2}}$ (Self-reciprocal).

(b) $\mathscr{H}_0\left[\delta(r - a)\right] = a\, J_0(a\kappa)$.

(c) $\mathscr{H}_0\left[\frac{1}{r}\right] = \frac{1}{\kappa}$.

24. Using the Parseval relation (7.3.2), show that

$$I(a,b) = \int_0^\infty \frac{1}{\kappa} J_{n+1}(a\kappa)\, J_{n+1}(b\kappa)\, d\kappa = \frac{1}{2(n+1)} \left(\frac{a}{b}\right)^{n+1},$$
$$0 < a < b, \;\; n + \frac{1}{2} > 0.$$

25. (a) Solve the axisymmetric Dirichlet problem in a half space described by Laplace equation

$$\begin{aligned} &u_{rr} + \frac{1}{r}u_r + u_{zz} = 0, \qquad 0 < r < \infty, \quad z > 0, \\ &u(r,0) = f(r), \qquad 0 < r < \infty, \\ &u(r,z) \to 0 \quad \text{as } r \to \infty, \;\; z \to \infty. \end{aligned}$$

(b) Find the solution of (a) when $f(r) = H(c - r)$.

(c) Find the solution of (a) when $f(r) = \frac{1}{\sqrt{r^2 + a^2}}$, $a > 0$.

26. Solve the axisymmetric small-amplitude vibration of a thin elastic plate governed by the equation

$$a^2 \left(\frac{\partial^2}{\partial r^2} + \frac{\partial}{r\,\partial r}\right)^2 u(r,t) + \frac{\partial^2 u}{\partial t^2} = 0, \qquad 0 < r < \infty, \;\; t > 0$$

with the initial conditions

$$u(r,0) = f(r), \quad u_t(r,0) = 0, \qquad 0 < r < \infty,$$

where $a = \frac{D}{2\rho h}$, D is the flexural rigidity, ρ is the density, and $2h$ is the thickness of the plate.

27. Solve the forced vibration problem of an elastic membrane described by the non-homogeneous boundary value problem

$$\begin{aligned} &u_{rr} + \frac{1}{r}u_r - \frac{1}{c^2}u_{tt} = -\frac{1}{T}p(r,t), \qquad 0 < r < \infty, \quad t > 0, \\ &u(r,0) = f(r), \quad u_t(r,0) = g(r), \qquad 0 < r < \infty, \\ &u(r,t) \text{is bounded at } \infty \;\; (r \to \infty), \end{aligned}$$

where T is the tension of the membrane and $c^2 = \frac{T}{\rho}$.