8

Mellin Transforms and Their Applications

> "One cannot understand ... the universality of laws of nature, the relationship of things, without an understanding of mathematics. There is no other way to do it."
>
> Richard P. Feynman

> "The research worker, in his efforts to express the fundamental laws of Nature in mathematical form, should strive mainly for mathematical beauty. He should take simplicity into consideration in a subordinate way to beauty. ... It often happens that the requirements of simplicity and beauty are the same, but where they clash the latter must take precedence."
>
> Paul Dirac

8.1 Introduction

This chapter deals with the theory and applications of the Mellin transform. We derive the Mellin transform and its inverse from the complex Fourier transform. This is followed by several examples and the basic operational properties of Mellin transforms. We discuss several applications of Mellin transforms to boundary value problems and to summation of infinite series. The Weyl transform and the Weyl fractional derivatives with examples are also included.

Historically, Riemann (1876) first recognized the *Mellin transform* in his famous memoir on prime numbers. Its explicit formulation was given by Cahen (1894). Almost simultaneously, Mellin (1896, 1902) gave an elaborate discussion of the Mellin transform and its inversion formula.

8.2 Definition of the Mellin Transform and Examples

We derive the Mellin transform and its inverse from the complex Fourier transform and its inverse, which are defined respectively by

$$\mathscr{F}\{g(\xi)\} = G(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ik\xi} g(\xi)d\xi, \tag{8.2.1}$$

$$\mathscr{F}^{-1}\{G(k)\} = g(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ik\xi} G(k)dk. \tag{8.2.2}$$

Making the changes of variables $\exp(\xi) = x$ and $ik = c - p$, where c is a constant, in results (8.2.1) and (8.2.2) we obtain

$$G(ip - ic) = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} x^{p-c-1} g(\log x)dx, \tag{8.2.3}$$

$$g(\log x) = \frac{1}{\sqrt{2\pi}} \int_{c-i\infty}^{c+i\infty} x^{c-p} G(ip - ic)dp. \tag{8.2.4}$$

We now write $\frac{1}{\sqrt{2\pi}} x^{-c} g(\log x) \equiv f(x)$ and $G(ip - ic) \equiv \tilde{f}(p)$ to define the *Mellin transform* of $f(x)$ and the *inverse Mellin transform* as

$$\mathscr{M}\{f(x)\} = \tilde{f}(p) = \int_{0}^{\infty} x^{p-1} f(x)dx, \tag{8.2.5}$$

$$\mathscr{M}^{-1}\{\tilde{f}(p)\} = f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} x^{-p} \tilde{f}(p)dp, \tag{8.2.6}$$

where $f(x)$ is a real valued function defined on $(0, \infty)$ and the Mellin transform variable p is a complex number. Sometimes, the Mellin transform of $f(x)$ is denoted explicitly by $\tilde{f}(p) = \mathscr{M}[f(x), p]$. Obviously, $\mathscr{M}$ and $\mathscr{M}^{-1}$ are linear integral operators.

Example 8.2.1 (a) If $f(x) = e^{-nx}$, where $n > 0$, then

$$\mathscr{M}\{e^{-nx}\} = \tilde{f}(p) = \int_{0}^{\infty} x^{p-1} e^{-nx} dx,$$

which is, by putting $nx = t$,

$$= \frac{1}{n^p} \int_0^\infty t^{p-1} e^{-t} dt = \frac{\Gamma(p)}{n^p}. \tag{8.2.7}$$

(b) If $f(x) = \dfrac{1}{1+x}$, then

$$\mathscr{M} \left\{ \frac{1}{1+x} \right\} = \tilde{f}(p) = \int_0^\infty x^{p-1} \cdot \frac{dx}{1+x},$$

which is, by substituting $x = \dfrac{t}{1-t}$ or $t = \dfrac{x}{1+x}$,

$$= \int_0^1 t^{p-1}(1-t)^{(1-p)-1} dt = B(p, 1-p) = \Gamma(p)\Gamma(1-p),$$

which is, by a well-known result for the gamma function,

$$= \pi \operatorname{cosec}(p\pi), \qquad 0 < \operatorname{Re}(p) < 1. \tag{8.2.8}$$

(c) If $f(x) = (e^x - 1)^{-1}$, then

$$\mathscr{M} \left\{ \frac{1}{e^x - 1} \right\} = \tilde{f}(p) = \int_0^\infty x^{p-1} \frac{1}{e^x - 1} dx,$$

which is, by using $\displaystyle\sum_{n=0}^\infty e^{-nx} = \frac{1}{1 - e^{-x}}$ and hence, $\displaystyle\sum_{n=1}^\infty e^{-nx} = \frac{1}{e^x - 1}$,

$$= \sum_{n=1}^\infty \int_0^\infty x^{p-1} e^{-nx} dx = \sum_{n=1}^\infty \frac{\Gamma(p)}{n^p} = \Gamma(p)\zeta(p), \tag{8.2.9}$$

where $\zeta(p) = \displaystyle\sum_{n=1}^\infty \frac{1}{n^p}$, $(\operatorname{Re} p > 1)$ is the famous *Riemann zeta function.*

(d) If $f(x) = \dfrac{2}{e^{2x} - 1}$, then

$$\begin{aligned} \mathscr{M} \left\{ \frac{2}{e^{2x} - 1} \right\} &= \tilde{f}(p) = 2 \int_0^\infty x^{p-1} \frac{dx}{e^{2x} - 1} = 2 \sum_{n=1}^\infty \int_0^\infty x^{p-1} e^{-2nx} dx \\ &= 2 \sum_{n=1}^\infty \frac{\Gamma(p)}{(2n)^p} = 2^{1-p}\,\Gamma(p) \sum_{n=1}^\infty \frac{1}{n^p} = 2^{1-p}\,\Gamma(p)\zeta(p). \end{aligned} \tag{8.2.10}$$

(e) If $f(x) = \dfrac{1}{e^x + 1}$, then

$$\mathscr{M}\left\{\frac{1}{e^x+1}\right\} = (1 - 2^{1-p})\Gamma(p)\,\zeta(p). \tag{8.2.11}$$

This follows from the result

$$\left[\frac{1}{e^x-1} - \frac{1}{e^x+1}\right] = \frac{2}{e^{2x}-1}$$

combined with (8.2.9) and (8.2.10).

(f) If $f(x) = \dfrac{1}{(1+x)^n}$, then

$$\mathscr{M}\left\{\frac{1}{(1+x)^n}\right\} = \int_0^\infty x^{p-1}(1+x)^{-n}\,dx,$$

which is, by putting $x = \dfrac{t}{1-t}$ or $t = \dfrac{x}{1+x}$,

$$= \int_0^1 t^{p-1}(1-t)^{n-p-1}\,dt$$

$$= B(p, n-p) = \frac{\Gamma(p)\Gamma(n-p)}{\Gamma(n)}, \tag{8.2.12}$$

where $B(p,q)$ is the standard beta function.
Hence,

$$\mathscr{M}^{-1}\{\Gamma(p)\Gamma(n-p)\} = \frac{\Gamma(n)}{(1+x)^n}.$$

(g) Find the Mellin transform of $\cos kx$ and $\sin kx$.
It follows from Example 8.2.1(a) that

$$\mathscr{M}\left[e^{-ikx}\right] = \frac{\Gamma(p)}{(ik)^p} = \frac{\Gamma(p)}{k^p}\left(\cos\frac{p\pi}{2} - i\sin\frac{p\pi}{2}\right).$$

Separating real and imaginary parts, we find

$$\mathscr{M}\left[\cos kx\right] = k^{-p}\,\Gamma(p)\cos\left(\frac{\pi p}{2}\right), \tag{8.2.13}$$

$$\mathscr{M}\left[\sin kx\right] = k^{-p}\,\Gamma(p)\sin\left(\frac{\pi p}{2}\right). \tag{8.2.14}$$

These results can be used to calculate the Fourier cosine and Fourier sine transforms of x^{p-1}. Result (8.2.13) can be written as

$$\int_0^\infty x^{p-1}\cos kx\,dx = \frac{\Gamma(p)}{k^p}\cos\left(\frac{\pi p}{2}\right).$$

Or, equivalently,

$$\mathscr{F}_c \left\{ \sqrt{\frac{\pi}{2}}\, x^{p-1} \right\} = \frac{\Gamma(p)}{k^p} \cos\left(\frac{\pi p}{2}\right).$$

Or,

$$\mathscr{F}_c \left\{x^{p-1}\right\} = \sqrt{\frac{2}{\pi}}\, \frac{\Gamma(p)}{k^p} \cos\left(\frac{\pi p}{2}\right). \tag{8.2.15}$$

Similarly,

$$\mathscr{F}_s \left\{x^{p-1}\right\} = \sqrt{\frac{2}{\pi}}\, \frac{\Gamma(p)}{k^p} \sin\left(\frac{\pi p}{2}\right). \tag{8.2.16}$$

$\square$

8.3 Basic Operational Properties of Mellin Transforms

If $\mathscr{M}\{f(x)\} = \tilde{f}(p)$, then the following operational properties hold:

(a) (*Scaling Property*).

$$\mathscr{M}\left\{f(ax)\right\} = a^{-p}\tilde{f}(p), \;\; a > 0. \tag{8.3.1}$$

PROOF By definition, we have,

$$\mathscr{M}\left\{f(ax)\right\} = \int_0^\infty x^{p-1} f(ax)\,dx,$$

which is, by substituting $ax = t$,

$$= \frac{1}{a^p} \int_0^\infty t^{p-1} f(t)\,dt = \frac{\tilde{f}(p)}{a^p}.$$

$\blacksquare$

(b) (*Shifting Property*).

$$\mathscr{M}\left[x^a\, f(x)\right] = \tilde{f}(p+a). \tag{8.3.2}$$

Its proof follows from the definition.

(c)
$$\mathscr{M}\left\{f(x^a)\right\} = \frac{1}{a}\, \tilde{f}\left(\frac{p}{a}\right), \tag{8.3.3}$$

$$\mathscr{M}\left\{\frac{1}{x}f\left(\frac{1}{x}\right)\right\}=\tilde{f}(1-p), \tag{8.3.4}$$

$$\mathscr{M}\left\{(\log x)^n f(x)\right\}=\frac{d^n}{dp^n}\tilde{f}(p), \quad n=1,2,3,\ldots. \tag{8.3.5}$$

The proofs of (8.3.3) and (8.3.4) are easy and hence, left to the reader.

Result (8.3.5) can easily be proved by using the result

$$\frac{d}{dp}x^{p-1}=(\log x)x^{p-1}. \tag{8.3.6}$$

(d) (*Mellin Transforms of Derivatives*).

$$\mathscr{M}\left[f'(x)\right]=-(p-1)\tilde{f}(p-1), \tag{8.3.7}$$

provided $[x^{p-1}f(x)]$ vanishes as $x\to 0$ and as $x\to\infty$.

$$\mathscr{M}\left[f''(x)\right]=(p-1)(p-2)\tilde{f}(p-2). \tag{8.3.8}$$

More generally,

$$\begin{aligned}\mathscr{M}\left[f^{(n)}(x)\right] &= (-1)^n\frac{\Gamma(p)}{\Gamma(p-n)}\tilde{f}(p-n)\\ &= (-1)^n\frac{\Gamma(p)}{\Gamma(p-n)}\mathscr{M}\left[f(x),p-n\right],\end{aligned} \tag{8.3.9}$$

provided $x^{p-r-1}f^{(r)}(x)=0$ as $x\to 0$ for $r=0,1,2,\ldots,(n-1)$.

PROOF We have, by definition,

$$\mathscr{M}\left[f'(x)\right]=\int_0^\infty x^{p-1}f'(x)\,dx,$$

which is, integrating by parts,

$$\begin{aligned}&=\left[x^{p-1}f(x)\right]_0^\infty-(p-1)\int_0^\infty x^{p-2}f(x)\,dx\\ &=-(p-1)\tilde{f}(p-1).\end{aligned}$$

∎

The proofs of (8.3.8) and (8.3.9) are similar and left to the reader.

(e) If $\mathscr{M}\left\{f(x)\right\}=\tilde{f}(p)$, then

$$\mathscr{M}\left\{xf'(x)\right\}=-p\tilde{f}(p), \tag{8.3.10}$$

provided $x^p f(x)$ vanishes at $x=0$ and as $x\to\infty$.

$$\mathscr{M}\{x^2 f''(x)\} = (-1)^2 p(p+1)\tilde{f}(p). \tag{8.3.11}$$

More generally,

$$\mathscr{M}\{x^n f^{(n)}(x)\} = (-1)^n \frac{\Gamma(p+n)}{\Gamma(p)}\tilde{f}(p). \tag{8.3.12}$$

PROOF We have, by definition,

$$\mathscr{M}\{xf'(x)\} = \int_0^\infty x^p f'(x)dx,$$

which is, integrating by parts,

$$= [x^p f(x)]_0^\infty - p\int_0^\infty x^{p-1} f(x)dx = -p\tilde{f}(p).$$

▮

Similar arguments can be used to prove results (8.3.11) and (8.3.12).

(f) (*Mellin Transforms of Differential Operators*).
If $\mathscr{M}\{f(x)\} = \tilde{f}(p)$, then

$$\mathscr{M}\left[\left(x\frac{d}{dx}\right)^2 f(x)\right] = \mathscr{M}\left[x^2 f''(x) + xf'(x)\right] = (-1)^2\, p^2 \tilde{f}(p), \tag{8.3.13}$$

and more generally,

$$\mathscr{M}\left[\left(x\frac{d}{dx}\right)^n f(x)\right] = (-1)^n p^n \tilde{f}(p). \tag{8.3.14}$$

PROOF We have, by definition,

$$\begin{aligned}\mathscr{M}\left[\left(x\frac{d}{dx}\right)^2 f(x)\right] &= \mathscr{M}\left[x^2 f''(x) + x\, f'(x)\right]\\ &= \mathscr{M}\left[x^2 f''(x)\right] + \mathscr{M}\left[x\, f'(x)\right]\\ &= -p\tilde{f}(p) + p(p+1)\tilde{f}(p) \qquad \text{by (8.3.10) and (8.3.11)}\\ &= (-1)^2\, p^2\, \tilde{f}(p).\end{aligned}$$

▮

Similar arguments can be used to prove the general result (8.3.14).

(g) (*Mellin Transforms of Integrals*).

$$\mathscr{M}\left\{\int_0^x f(t)dt\right\} = -\frac{1}{p}\,\tilde{f}(p+1). \tag{8.3.15}$$

In general,

$$\mathscr{M}\{I_n\, f(x)\} = \mathscr{M}\left\{\int_0^x I_{n-1} f(t)dt\right\} = (-1)^n \frac{\Gamma(p)}{\Gamma(p+n)}\tilde{f}(p+n), \tag{8.3.16}$$

where $I_n\, f(x)$ is the nth repeated integral of $f(x)$ defined by

$$I_n f(x) = \int_0^x I_{n-1} f(t)dt. \tag{8.3.17}$$

PROOF We write

$$F(x) = \int_0^x f(t)dt$$

so that $F'(x) = f(x)$ with $F(0) = 0$. Application of (8.3.7) with $F(x)$ as defined gives

$$\mathscr{M}\,\{f(x) = F'(x), p\} = -(p-1)\mathscr{M}\left\{\int_0^x f(t)dt,\, p-1\right\},$$

which is, replacing p by $p+1$,

$$\mathscr{M}\left\{\int_0^x f(t)\,dt,\, p\right\} = -\frac{1}{p}\mathscr{M}\,\{f(x), p+1\} = -\frac{1}{p}\,\tilde{f}(p+1).$$

An argument similar to this can be used to prove (8.3.16). ∎

(h) (*Convolution Type Theorems*).
If $\mathscr{M}\,\{f(x)\} = \tilde{f}(p)$ and $\mathscr{M}\{g(x)\} = \tilde{g}(p)$, then

$$\mathscr{M}\,[f(x) * g(x)] = \mathscr{M}\left[\int_0^\infty f(\xi)\, g\left(\frac{x}{\xi}\right)\frac{d\xi}{\xi}\right] = \tilde{f}(p)\tilde{g}(p), \tag{8.3.18}$$

$$\mathscr{M}\,[f(x) \circ g(x)] = \mathscr{M}\left[\int_0^\infty f(x\xi)\, g(\xi)d\xi\right] = \tilde{f}(p)\tilde{g}(1-p). \tag{8.3.19}$$

PROOF We have, by definition,

$$\begin{aligned}
\mathscr{M}\left[f(x)*g(x)\right] &= \mathscr{M}\left[\int_0^\infty f(\xi)\, g\left(\frac{x}{\xi}\right)\frac{d\xi}{\xi}\right] \\
&= \int_0^\infty x^{p-1}dx \int_0^\infty f(\xi)\, g\left(\frac{x}{\xi}\right)\frac{d\xi}{\xi} \\
&= \int_0^\infty f(\xi)\,\frac{d\xi}{\xi}\int_0^\infty x^{p-1} g\left(\frac{x}{\xi}\right)\,dx, \quad \left(\frac{x}{\xi}=\eta\right), \\
&= \int_0^\infty f(\xi)\,\frac{d\xi}{\xi}\int_0^\infty (\xi\eta)^{p-1} g(\eta)\,\xi\,d\eta \\
&= \int_0^\infty \xi^{p-1} f(\xi)d\xi \int_0^\infty \eta^{p-1} g(\eta)d\eta = \tilde{f}(p)\tilde{g}(p).
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
\mathscr{M}\left[f(x)\circ g(x)\right] &= \mathscr{M}\left[\int_0^\infty f(x\xi)\, g(\xi)d\xi\right] \\
&= \int_0^\infty x^{p-1}dx \int_0^\infty f(x\xi)\, g(\xi)d\xi, \quad (x\xi=\eta), \\
&= \int_0^\infty g(\xi)d\xi \int_0^\infty \eta^{p-1}\xi^{1-p} f(\eta)\frac{d\eta}{\xi} \\
&= \int_0^\infty \xi^{1-p-1} g(\xi)d\xi \int_0^\infty \eta^{p-1} f(\eta)d\eta = \tilde{g}(1-p)\tilde{f}(p).
\end{aligned}$$

∎

Note that, in this case, the operation $\circ$ is not commutative.
Clearly, putting $x=s$,

$$\mathscr{M}^{-1}\{\tilde{f}(1-p)\tilde{g}(p)\} = \int_0^\infty g(st)f(t)dt.$$

Putting $g(t)=e^{-t}$ and $\tilde{g}(p)=\Gamma(p)$, we obtain the Laplace transform of $f(t)$

$$\mathscr{M}^{-1}\{\tilde{f}(1-p)\Gamma(p)\} = \int_0^\infty e^{-st}f(t)dt = \mathscr{L}\{f(t)\} = \bar{f}(s). \tag{8.3.20}$$

(i) (*Parseval's Type Property*).
If $\mathscr{M}\{f(x)\} = \tilde{f}(p)$ and $\mathscr{M}\{g(x)\} = \tilde{g}(p)$, then

$$\mathscr{M}[f(x)g(x)] = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \tilde{f}(s)\tilde{g}(p-s)ds. \tag{8.3.21}$$

Or, equivalently,

$$\int_0^\infty x^{p-1} f(x)g(x)dx = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \tilde{f}(s)\tilde{g}(p-s)ds. \tag{8.3.22}$$

In particular, when $p = 1$, we obtain the *Parseval formula* for the Mellin transform,

$$\int_0^\infty f(x)g(x)dx = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \tilde{f}(s)\tilde{g}(1-s)ds. \tag{8.3.23}$$

PROOF By definition, we have

$$\begin{aligned}
\mathscr{M}[f(x)g(x)] &= \int_0^\infty x^{p-1} f(x)g(x)dx \\
&= \frac{1}{2\pi i}\int_0^\infty x^{p-1} g(x)dx \int_{c-i\infty}^{c+i\infty} x^{-s}\tilde{f}(s)ds \\
&= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \tilde{f}(s)ds \int_0^\infty x^{p-s-1} g(x)dx \\
&= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \tilde{f}(s)\tilde{g}(p-s)ds.
\end{aligned}$$

When $p = 1$, the above result becomes (8.3.23). ∎

8.4 Applications of Mellin Transforms

Example 8.4.1 Obtain the solution of the boundary value problem

$$x^2 u_{xx} + x u_x + u_{yy} = 0, \qquad 0 \le x < \infty, \qquad 0 < y < 1 \tag{8.4.1}$$

$$u(x,0) = 0, \quad u(x,1) = \left\{ \begin{array}{ll} A, & 0 \le x \le 1 \\ 0, & x > 1 \end{array} \right\}, \tag{8.4.2}$$

where A is a constant.

We apply the Mellin transform of $u(x,\, y)$ with respect to x defined by

$$\tilde{u}(p,y) = \int_0^\infty x^{p-1} u(x,y)\, dx$$

to reduce the given system into the form

$$\tilde{u}_{yy} + p^2 \tilde{u} = 0, \quad 0 < y < 1$$

$$\tilde{u}(p,0) = 0, \quad \tilde{u}(p,1) = A \int_0^1 x^{p-1} dx = \frac{A}{p}.$$

The solution of the transformed problem is

$$\tilde{u}(p,y) = \frac{A}{p} \frac{\sin\, py}{\sin\, p}, \qquad 0 < \,\mathrm{Re}\, p < 1.$$

The inverse Mellin transform gives

$$u(x,y) = \frac{A}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{x^{-p}}{p} \frac{\sin\, py}{\sin\, p}\, dp, \tag{8.4.3}$$

where $\tilde{u}(p,y)$ is analytic in the vertical strip $0 < \mathrm{Re}\,(p) = c < \pi$. The integrand of (8.4.3) has simple poles at $p = n\pi$, $n = 1, 2, 3, \ldots$ which lie inside a semi-circular contour in the right half plane. Evaluating (8.4.3) by theory of residues gives the solution for $x > 1$ as

$$u(x,y) = \frac{A}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} (-1)^n \, x^{-n\pi} \sin\, n\pi y. \tag{8.4.4}$$

$\square$

Example 8.4.2 (*Potential in an Infinite Wedge*).
Find the potential $\phi(r,\theta)$ that satisfies the Laplace equation

$$r^2\phi_{rr} + r\phi_r + \phi_{\theta\theta} = 0 \tag{8.4.5}$$

in an infinite wedge $0 < r < \infty$, $-\alpha < \theta < \alpha$ as shown in Figure 8.1 with the boundary conditions

$$\phi(r,\alpha) = f(r), \;\; \phi(r,-\alpha) = g(r) \quad 0 \le r < \infty, \tag{8.4.6ab}$$

$$\phi(r,\theta) \to 0 \;\text{ as }\; r \to \infty \;\text{ for all }\; \theta \;\text{ in }\; -\alpha < \theta < \alpha. \tag{8.4.7}$$

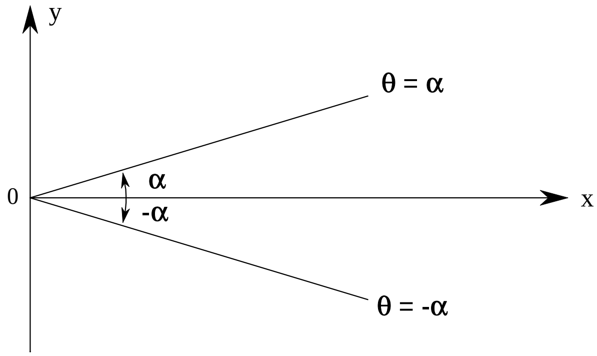

Figure 8.1 An infinite wedge.

We apply the Mellin transform of the potential $\phi(r,\theta)$ defined by

$$\mathscr{M}\left[\phi(r,\theta)\right] = \tilde{\phi}(p,\theta) = \int_0^\infty r^{p-1}\phi(r,\theta)\,dr$$

to the differential system (8.4.5)–(8.4.7) to obtain

$$\frac{d^2\tilde{\phi}}{d\theta^2} + p^2\tilde{\phi} = 0, \tag{8.4.8}$$

$$\tilde{\phi}(p,\alpha) = \tilde{f}(p), \qquad \tilde{\phi}(p,-\alpha) = \tilde{g}(p). \tag{8.4.9ab}$$

The general solution of the transformed equation is

$$\tilde{\phi}(p,\theta) = A\cos p\theta + B\sin p\theta, \tag{8.4.10}$$

where A and B are functions of p and α. The boundary conditions (8.4.9ab) determine A and B, which satisfy

$$A\,\cos\,p\alpha + B\,\sin\,p\alpha = \tilde{f}(p),$$
$$A\,\cos\,p\alpha - B\,\sin\,p\alpha = \tilde{g}(p).$$

These give $\quad A = \dfrac{\tilde{f}(p)+\tilde{g}(p)}{2\cos p\alpha}, \quad B = \dfrac{\tilde{f}(p)-\tilde{g}(p)}{2\sin p\alpha}.$

Thus, solution (8.4.10) becomes

$$\begin{aligned}\tilde{\phi}(p,\theta) &= \tilde{f}(p).\frac{\sin p(\alpha+\theta)}{\sin(2\,p\alpha)} + \tilde{g}(p)\frac{\sin p(\alpha-\theta)}{\sin(2\,p\alpha)} \\ &= \tilde{f}(p)\tilde{h}(p,\alpha+\theta) + \tilde{g}(p)\tilde{h}(p,\alpha-\theta),\end{aligned} \tag{8.4.11}$$

where

$$\tilde{h}(p,\theta) = \frac{\sin p\theta}{\sin(2\,p\alpha)}.$$

Or, equivalently,

$$h(r,\theta) = \mathscr{M}^{-1}\left\{\frac{\sin p\theta}{\sin 2\,p\alpha}\right\} = \left(\frac{1}{2\,\alpha}\right)\frac{r^n \sin n\theta}{(1+2\,r^n\cos n\theta + r^{2n})}, \tag{8.4.12}$$

where

$$n = \frac{\pi}{2\alpha} \quad \text{or,} \quad 2\alpha = \frac{\pi}{n}.$$

Application of the inverse Mellin transform to (8.4.11) gives

$$\phi(r,\theta) = \mathscr{M}^{-1}\left\{\tilde{f}(p)\tilde{h}(p,\alpha+\theta)\right\} + \mathscr{M}^{-1}\left\{\tilde{g}(p)\tilde{h}(p,\alpha-\theta)\right\},$$

which is, by the convolution property (8.3.18),

$$\begin{aligned}\phi(r,\theta) = \frac{r^n\cos n\theta}{2\alpha}&\left[\int_0^\infty \frac{\xi^{n-1}f(\xi)d\xi}{\xi^{2n}-2(r\xi)^n\sin n\theta + r^{2n}}\right. \\ &\left.+\int_0^\infty \frac{\xi^{n-1}g(\xi)d\xi}{\xi^{2n}+2(r\xi)^n\sin n\theta + r^{2n}}\right], \quad |\alpha| < \frac{\pi}{2n}.\end{aligned} \tag{8.4.13}$$

This is the formal solution of the problem.

In particular, when $f(r) = g(r)$, solution (8.4.11) becomes

$$\tilde{\phi}(p,\theta) = \tilde{f}(p)\,\frac{\cos p\theta}{\cos p\alpha} = \tilde{f}(p)\tilde{h}(p,\theta), \tag{8.4.14}$$

where

$$\tilde{h}(p,\theta) = \frac{\cos p\theta}{\cos p\alpha} = \mathscr{M}\{h(r,\theta)\}.$$

Application of the inverse Mellin transform to (8.4.14) combined with the convolution property (8.3.18) yields the solution

$$\phi(r,\theta) = \int_0^\infty f(\xi)\, h\left(\frac{r}{\xi},\theta\right) \frac{d\xi}{\xi}, \tag{8.4.15}$$

where

$$h(r,\theta) = \mathscr{M}^{-1}\left\{\frac{\cos p\theta}{\cos p\alpha}\right\} = \left(\frac{r^n}{\alpha}\right) \frac{(1+r^{2n})\cos(n\theta)}{(1+2r^{2n}\cos 2n\theta + r^{2n})}, \tag{8.4.16}$$

and $n = \dfrac{\pi}{2\alpha}$. $\square$

Some applications of the Mellin transform to boundary value problems are given by Sneddon (1951) and Tranter (1966).

Example 8.4.3 Solve the integral equation

$$\int_0^\infty f(\xi)\, k(x\xi) d\xi = g(x), \qquad x > 0. \tag{8.4.17}$$

Application of the Mellin transform with respect to x to equation (8.4.17) combined with (8.3.19) gives

$$\tilde{f}(1-p)\tilde{k}(p) = \tilde{g}(p),$$

which gives, replacing p by $1-p$,

$$\tilde{f}(p) = \tilde{g}(1-p)\tilde{h}(p),$$

where

$$\tilde{h}(p) = \frac{1}{\tilde{k}(1-p)}.$$

The inverse Mellin transform combined with (8.3.19) leads to the solution

$$f(x) = \mathscr{M}^{-1}\left\{\tilde{g}(1-p)\tilde{h}(p)\right\} = \int_0^\infty g(\xi) h(x\xi) d\xi, \tag{8.4.18}$$

provided $h(x) = \mathscr{M}^{-1}\left\{\tilde{h}(p)\right\}$ exists. Thus, the problem is formally solved.

If, in particular, $\tilde{h}(p) = \tilde{k}(p)$, then the solution of (8.4.18) becomes

$$f(x) = \int_0^\infty g(\xi)\, k(x\xi) d\xi, \tag{8.4.19}$$

provided $\tilde{k}(p)\tilde{k}(1-p)=1$. ▯

Example 8.4.4 Solve the integral equation

$$\int_0^\infty f(\xi)\, g\left(\frac{x}{\xi}\right)\frac{d\xi}{\xi} = h(x), \qquad (8.4.20)$$

where $f(x)$ is unknown and $g(x)$ and $h(x)$ are given functions.

Applications of the Mellin transform with respect to x gives

$$\tilde{f}(p)=\tilde{h}(p)\tilde{k}(p), \qquad \tilde{k}(p)=\frac{1}{\tilde{g}(p)}.$$

Inversion, by the convolution property (8.3.18), gives the solution

$$f(x)=\mathscr{M}^{-1}\left\{\tilde{h}(p)\tilde{k}(p)\right\}=\int_0^\infty h(\xi)\, k\left(\frac{x}{\xi}\right)\frac{d\xi}{\xi}. \qquad (8.4.21)$$

▯

8.5 Mellin Transforms of the Weyl Fractional Integral and the Weyl Fractional Derivative

DEFINITION 8.5.1 *The Mellin transform of the Weyl fractional integral of $f(x)$ is defined by*

$$W^{-\alpha}[f(x)]=\frac{1}{\Gamma(\alpha)}\int_x^\infty (t-x)^{\alpha-1}f(t)dt, \quad 0<\operatorname{Re}\alpha<1, \quad x>0. \qquad (8.5.1)$$

Often ${}_xW_\infty^{-\alpha}$ is used instead of $W^{-\alpha}$ to indicate the limits to integration. Result (8.5.1) can be interpreted as the Weyl transform of $f(t)$, defined by

$$W^{-\alpha}[f(t)]=F(x,\alpha)=\frac{1}{\Gamma(\alpha)}\int_x^\infty (t-x)^{\alpha-1}f(t)dt. \qquad (8.5.2)$$

We first give some simple examples of the Weyl transform.

If $f(t)=\exp(-at)$, Re $a>0$, then the Weyl transform of $f(t)$ is given by

$$W^{-\alpha}[\exp(-at)]=\frac{1}{\Gamma(\alpha)}\int_x^\infty (t-x)^{\alpha-1}\exp(-at)dt,$$

which is, by the change of variable $t-x=y$,

$$= \frac{e^{-ax}}{\Gamma(\alpha)} \int_0^\infty y^{\alpha-1} \exp(-ay)dy$$

which is, by letting $ay=t$,

$$W^{-\alpha}[f(t)] = \frac{e^{-ax}}{a^\alpha} \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} e^{-t} dt = \frac{e^{-ax}}{a^\alpha}. \tag{8.5.3}$$

Similarly, it can be shown that

$$W^{-\alpha}[t^{-\mu}] = \frac{\Gamma(\mu-\alpha)}{\Gamma(\mu)} x^{\alpha-\mu}, \qquad 0 < \operatorname{Re}\alpha < \operatorname{Re}\mu. \tag{8.5.4}$$

Making reference to Gradshteyn and Ryzhik (2000, p. 424), we obtain

$$W^{-\alpha}[\sin at] = a^{-\alpha} \sin\left(ax + \frac{\pi\alpha}{2}\right), \tag{8.5.5}$$

$$W^{-\alpha}[\cos at] = a^{-\alpha} \cos\left(ax + \frac{\pi\alpha}{2}\right), \tag{8.5.6}$$

where $0 < \operatorname{Re}\alpha < 1$ and $a > 0$.

It can be shown that, for any two positive numbers α and β, the Weyl fractional integral satisfies the laws of exponents

$$W^{-\alpha}[W^{-\beta} f(x)] = W^{-(\beta+\alpha)}[f(x)] = W^{-\beta}[W^{-\alpha} f(x)]. \tag{8.5.7}$$

Invoking a change of variable $t-x=y$ in (8.5.1), we obtain

$$W^{-\alpha}[f(x)] = \frac{1}{\Gamma(\alpha)} \int_0^\infty y^{\alpha-1} f(x+y)dy. \tag{8.5.8}$$

We next differentiate (8.5.8) to obtain, $D = \dfrac{d}{dx}$,

$$\begin{aligned} D[W^{-\alpha} f(x)] &= \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} \frac{\partial}{\partial x} f(x+t)dt \\ &= \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} Df(x+t)dt \\ &= W^{-\alpha}[Df(x)]. \end{aligned} \tag{8.5.9}$$

A similar argument leads to a more general result

$$D^n[W^{-\alpha} f(x)] = W^{-\alpha}[D^n f(x)], \tag{8.5.10}$$

where n is a positive integer.

Or, symbolically,

$$D^n W^{-\alpha} = W^{-\alpha} D^n. \tag{8.5.11}$$

We now calculate the Mellin transform of the Weyl fractional integral by putting $h(t) = t^\alpha f(t)$ and $g\left(\frac{x}{t}\right) = \frac{1}{\Gamma(\alpha)}\left(1 - \frac{x}{t}\right)^{\alpha-1} H\left(1 - \frac{x}{t}\right)$, where $H\left(1 - \frac{x}{t}\right)$ is the Heaviside unit step function so that (8.5.1) becomes

$$F(x,\alpha) = \int_0^\infty h(t)\, g\left(\frac{x}{t}\right) \frac{dt}{t}, \tag{8.5.12}$$

which is, by the convolution property (8.3.18),

$$\tilde{F}(p,\alpha) = \tilde{h}(p)\tilde{g}(p),$$

where

$$\tilde{h}(p) = \mathscr{M}\{x^\alpha f(x)\} = \tilde{f}(p+\alpha),$$

and

$$\begin{aligned}\tilde{g}(p) &= \mathscr{M}\left\{\frac{1}{\Gamma(\alpha)}(1-x)^{\alpha-1} H(1-x)\right\}\\ &= \frac{1}{\Gamma(\alpha)}\int_0^1 x^{p-1}(1-x)^{\alpha-1}dx = \frac{B(p,\alpha)}{\Gamma(\alpha)} = \frac{\Gamma(p)}{\Gamma(p+\alpha)}.\end{aligned}$$

Consequently,

$$\tilde{F}(p,\alpha) = \mathscr{M}\left[W^{-\alpha} f(x), p\right] = \frac{\Gamma(p)}{\Gamma(p+\alpha)}\, \tilde{f}(p+\alpha). \tag{8.5.13}$$

It is important to note that this result is an obvious extension of result 7(b) in Exercise 8.8

DEFINITION 8.5.2 *If β is a positive number and n is the smallest integer greater than β such that $n - \beta = \alpha > 0$, the Weyl fractional derivative of a function $f(x)$ is defined by*

$$\begin{aligned}W^\beta[f(x)] &= E^n\, W^{-(n-\beta)}[f(x)]\\ &= \frac{(-1)^n}{\Gamma(n-\beta)} \frac{d^n}{dx^n} \int_x^\infty (t-x)^{n-\beta-1} f(t)dt,\end{aligned} \tag{8.5.14}$$

where $E = -D$.

Or, symbolically,

$$W^\beta = E^n W^{-\alpha} = E^n W^{-(n-\beta)}. \tag{8.5.15}$$

It can be shown that, for any β,

$$W^{-\beta}W^{\beta}=I=W^{\beta}W^{-\beta}. \tag{8.5.16}$$

And, for any β and γ, the Weyl fractional derivative satisfies the laws of exponents

$$W^{\beta}[W^{\gamma}f(x)]=W^{\beta+\gamma}[f(x)]=W^{\gamma}[W^{\beta}f(x)]. \tag{8.5.17}$$

We now calculate the Weyl fractional derivative of some elementary functions.

If $f(x)=\exp(-ax)$, $a>0$, then the definition (8.5.14) gives

$$W^{\beta}e^{-ax}=E^{n}[W^{-(n-\beta)}e^{-ax}]. \tag{8.5.18}$$

Writing $n-\beta=\alpha>0$ and using (8.5.3) yields

$$\begin{aligned} W^{\beta}e^{-ax} &= E^{n}[W^{-\alpha}e^{-ax}]=E^{n}[a^{-\alpha}e^{-ax}] \\ &= a^{-\alpha}(a^{n}e^{-ax})=a^{\beta}e^{-ax}. \end{aligned} \tag{8.5.19}$$

Replacing β by $-\alpha$ in (8.5.19) leads to result (8.5.3) as expected.

Similarly, we obtain

$$W^{\beta}x^{-\mu}=\frac{\Gamma(\beta+\mu)}{\Gamma(\mu)}\,x^{-(\beta+\mu)}. \tag{8.5.20}$$

It is easy to see that

$$W^{\beta}(\cos ax)=E[W^{-(1-\beta)}\cos ax],$$

which is, by (8.5.6),

$$=a^{\beta}\cos\left(ax-\frac{1}{2}\pi\beta\right). \tag{8.5.21}$$

Similarly,

$$W^{\beta}(\sin ax)=a^{\beta}\sin\left(ax-\frac{1}{2}\pi\beta\right), \tag{8.5.22}$$

provided α and β lie between 0 and 1.

If β is replaced by $-\alpha$, results (8.5.20)–(8.5.22) reduce to (8.5.4)–(8.5.6), respectively.

Finally, we calculate the Mellin transform of the Weyl fractional derivative with the help of (8.3.9) and find

$$\begin{aligned} \mathscr{M}\,[W^{\beta}f(x)] &= \mathscr{M}\,[E^{n}W^{-(n-\beta)}f(x)]=(-1)^{n}\mathscr{M}\,[D^{n}W^{-(n-\beta)}f(x)] \\ &= \frac{\Gamma(p)}{\Gamma(p-n)}\mathscr{M}\,[W^{-(n-\beta)}f(x),\,p-n], \end{aligned}$$

which is, by result (8.5.13),

$$= \frac{\Gamma(p)}{\Gamma(p-n)} \cdot \frac{\Gamma(p-n)}{\Gamma(p-\beta)} \tilde{f}(p-\beta)$$
$$= \frac{\Gamma(p)}{\Gamma(p-\beta)} \mathscr{M}[f(x), p-\beta]$$
$$= \frac{\Gamma(p)}{\Gamma(p-\beta)} \tilde{f}(p-\beta). \tag{8.5.23}$$

Example 8.5.1 (*The Fourier Transform of the Weyl Fractional Integral*).

$$\mathscr{F}\{W^{-\alpha} f(x)\} = \exp\left(-\frac{\pi i \alpha}{2}\right) k^{-\alpha} \mathscr{F}\{f(x)\}. \tag{8.5.24}$$

We have, by definition,

$$\mathscr{F}\{W^{-\alpha} f(x)\} = \frac{1}{\sqrt{2\pi}} \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{\infty} e^{-ikx} dx \int_{x}^{\infty} (t-x)^{\alpha-1} f(t) dt$$
$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) dt \cdot \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{t} \exp(-ikx)(t-x)^{\alpha-1} dx.$$

Thus,

$$\mathscr{F}\{W^{-\alpha} f(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikt} f(t) dt \cdot \frac{1}{\Gamma(\alpha)} \int_{0}^{\infty} e^{ik\tau} \tau^{\alpha-1} d\tau, \quad (t-x=\tau)$$
$$= \mathscr{F}\{f(x)\} \frac{1}{\Gamma(\alpha)} \mathscr{M}\{e^{ik\tau}\}$$
$$= \exp\left(-\frac{\pi i \alpha}{2}\right) k^{-\alpha} \mathscr{F}\{f(x)\}.$$

In the limit as $\alpha \to 0$

$$\lim_{\alpha \to 0} \mathscr{F}\{W^{-\alpha} f(x)\} = \mathscr{F}\{f(x)\}.$$

This implies that

$$W^0\{f(x)\} = f(x).$$

We conclude this section by proving a general property of the Riemann-Liouville fractional integral operator $D^{-\alpha}$, and the Weyl fractional integral operator $W^{-\alpha}$. It follows from the definition (6.2.1) that $D^{-\alpha} f(t)$ can be expressed as the convolution

$$D^{-\alpha} f(x) = g_\alpha(t) * f(t), \tag{8.5.25}$$

where

$$g_\alpha(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}, \ \ t > 0.$$

Similarly, $W^{-\alpha}f(x)$ can also be written in terms of the convolution

$$W^{-\alpha}f(x) = g_\alpha(-x) * f(x). \tag{8.5.26}$$

Then, under suitable conditions,

$$\mathscr{M}\left[D^{-\alpha}f(x)\right] = \frac{\Gamma(1-\alpha-p)}{\Gamma(1-p)}\,\tilde{f}(p+\alpha), \tag{8.5.27}$$

$$\mathscr{M}\left[W^{-\alpha}f(x)\right] = \frac{\Gamma(p)}{\Gamma(\alpha+p)}\,\tilde{f}(p+\alpha). \tag{8.5.28}$$

Finally, a formal computation gives

$$\begin{aligned}
\int_0^\infty \{D^{-\alpha}f(x)\}g(x)dx &= \frac{1}{\Gamma(\alpha)}\int_0^\infty g(x)dx\int_0^x (x-t)^{\alpha-1}f(t)dt \\
&= \int_0^\infty f(t)dt \cdot \frac{1}{\Gamma(\alpha)}\int_t^\infty (x-t)^{\alpha-1}g(x)dx \\
&= \int_0^\infty f(t)\left[W^{-\alpha}g(t)\right]dt,
\end{aligned}$$

which is, using the inner product notation,

$$\langle D^{-\alpha}f,\, g\rangle = \langle f,\, W^{-\alpha}g\rangle. \tag{8.5.29}$$

This show that $D^{-\alpha}$ and $W^{-\alpha}$ behave like adjoint operators. Obviously, this result can be used to define fractional integrals of distributions. This result is taken from Debnath and Grum (1988). $\square$

8.6 Application of Mellin Transforms to Summation of Series

In this section we discuss a method of summation of series that is particularly associated with the work of Macfarlane (1949).

THEOREM 8.6.1 If $\mathscr{M}\{f(x)\} = \tilde{f}(p)$**,** then

$$\sum_{n=0}^\infty f(n+a) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \tilde{f}(p)\,\xi(p,a)dp, \tag{8.6.1}$$

where $\xi(p,a)$ is the *Hurwitz zeta function* defined by

$$\xi(p,a)=\sum_{n=0}^{\infty}\frac{1}{(n+a)^p}, \qquad 0\le a\le 1,\ \mathrm{Re}(p)>1. \tag{8.6.2}$$

PROOF If follows from the inverse Mellin transform that

$$f(n+a)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\tilde{f}(p)(n+a)^{-p}\,dp. \tag{8.6.3}$$

Summing this over all n gives

$$\sum_{n=0}^{\infty}f(n+a)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\tilde{f}(p)\,\xi(p,a)\,dp.$$

This completes the proof.

Similarly, the scaling property (8.3.1) gives

$$f(nx)=\mathscr{M}^{-1}\{n^{-p}\,\tilde{f}(p)\}=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}x^{-p}\,n^{-p}\tilde{f}(p)dp.$$

Thus,

$$\sum_{n=1}^{\infty}f(nx)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}x^{-p}\tilde{f}(p)\,\zeta(p)dp=\mathscr{M}^{-1}\{\tilde{f}(p)\,\zeta(p)\}, \tag{8.6.4}$$

where $\zeta(p)=\sum_{n=1}^{\infty}n^{-p}$ is the *Riemann zeta function.*

When $x=1$, result (8.6.4) reduces to

$$\sum_{n=1}^{\infty}f(n)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\tilde{f}(p)\,\zeta(p)dp. \tag{8.6.5}$$

This can be obtained from (8.6.1) when $a=0$. ∎

Example 8.6.1 Show that

$$\sum_{n=1}^{\infty}(-1)^{n-1}n^{-p}=(1-2^{1-p})\,\zeta(p). \tag{8.6.6}$$

Using Example 8.2.1(a), we can write the left-hand side of (8.6.6) multiplied by t^n as

$$\begin{aligned}\sum_{n=1}^{\infty}(-1)^{n-1}n^{-p}t^n &= \sum_{n=1}^{\infty}(-1)^{n-1}t^n \cdot \frac{1}{\Gamma(p)}\int_0^{\infty} x^{p-1}e^{-nx}dx \\ &= \frac{1}{\Gamma(p)}\int_0^{\infty} x^{p-1}dx \sum_{n=1}^{\infty}(-1)^{n-1}t^{nx}e^{-nx} \\ &= \frac{1}{\Gamma(p)}\int_0^{\infty} x^{p-1}\cdot\frac{te^{-x}}{1+te^{-x}}\cdot dx \\ &= \frac{1}{\Gamma(p)}\int_0^{\infty} x^{p-1}\cdot\frac{t}{e^x+t}\,dx.\end{aligned}$$

In the limit as $t \to 1$, the above result gives

$$\begin{aligned}\sum_{n=1}^{\infty}(-1)^{n-1}n^{-p} &= \frac{1}{\Gamma(p)}\int_0^{\infty} x^{p-1}\frac{1}{e^x+1}\,dx \\ &= \frac{1}{\Gamma(p)}\mathscr{M}\left\{\frac{1}{e^x+1}\right\} = (1-2^{1-p})\,\zeta(p),\end{aligned}$$

in which result (8.2.11) is used. $\square$

Example 8.6.2 Show that

$$\sum_{n=1}^{\infty}\left(\frac{\sin an}{n}\right) = \frac{1}{2}(\pi - a), \qquad 0 < a < 2\pi. \tag{8.6.7}$$

The Mellin transform of $f(x) = \left(\dfrac{\sin ax}{x}\right)$ gives

$$\begin{aligned}\mathscr{M}\left[\frac{\sin ax}{x}\right] &= \int_0^{\infty} x^{p-2}\sin ax\,dx \\ &= \mathscr{F}_s\left\{\sqrt{\frac{\pi}{2}}\,x^{p-2}\right\} \\ &= -\frac{\Gamma(p-1)}{a^{p-1}}\cos\left(\frac{\pi p}{2}\right).\end{aligned}$$

Substituting this result into (8.6.5) gives

$$\sum_{n=1}^{\infty}\left(\frac{\sin an}{n}\right) = -\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\Gamma(p-1)}{a^{p-1}}\,\zeta(p)\,\cos\left(\frac{\pi p}{2}\right)\,dp. \tag{8.6.8}$$

We next use the well-known functional equation for the zeta function

$$(2\pi)^p\,\zeta(1-p) = 2\Gamma(p)\,\zeta(p)\cos\left(\frac{\pi p}{2}\right) \tag{8.6.9}$$

in the integrand of (8.6.8) to obtain

$$\sum_{n=1}^{\infty}\left(\frac{\sin an}{n}\right) = -\frac{a}{2}\cdot\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left(\frac{2\pi}{a}\right)^p\frac{\zeta(1-p)}{p-1}\,dp.$$

The integral has two simple poles at $p=0$ and $p=1$ with residues 1 and $-\pi/a$, respectively, and the complex integral is evaluated by calculating the residues at these poles. Thus, the sum of the series is

$$\sum_{n=1}^{\infty}\left(\frac{\sin an}{n}\right) = \frac{1}{2}(\pi - a).$$

$\square$

8.7 Generalized Mellin Transforms

In order to extend the applicability of the classical Mellin transform, Naylor (1963) generalized the method of Mellin integral transforms. This generalized Mellin transform is useful for finding solutions of boundary value problems in regions bounded by the natural coordinate surfaces of a spherical or cylindrical coordinate system. They can be used to solve boundary value problems in finite regions or in infinite regions bounded internally.

The *generalized Mellin transform* of a function $f(r)$ defined in $a<r<\infty$ is introduced by the integral

$$\mathscr{M}_-\{f(r)\} = F_-(p) = \int_a^\infty\left(r^{p-1} - \frac{a^{2p}}{r^{p+1}}\right)f(r)\,dr. \tag{8.7.1}$$

The inverse transform is given by

$$\mathscr{M}_-^{-1}\{F_-(p)\} = f(r) = \frac{1}{2\pi i}\int_L r^{-p}\,F(p)\,dp, \quad r>a, \tag{8.7.2}$$

where L is the line $\mathrm{Re}\,p=c$, and $F(p)$ is analytic in the strip $|\mathrm{Re}(p)| = |c| < \gamma$.

By integrating by parts, we can show that

$$\mathscr{M}_-\left[r^2\frac{\partial^2 f}{\partial r^2} + r\frac{\partial f}{\partial r}\right] = p^2\,F_-(p) + 2\,p\,a^p f(a), \tag{8.7.3}$$

provided $f(r)$ is appropriately behaved at infinity. More precisely,

$$\lim_{r\to\infty}\left[(r^p - a^{2p}\, r^{-p}) r f_r - p(r^p + a^{2p} r^{-p}) f\right] = 0. \tag{8.7.4}$$

Obviously, this generalized transform seems to be very useful for finding the solution of boundary value problems in which $f(r)$ is prescribed on the internal boundary at $r = a$.

On the other hand, if the derivative of $f(r)$ is prescribed at $r = a$, it is convenient to define the associated integral transform by

$$\mathscr{M}_+[f(r)] = F_+(p) = \int_a^\infty \left(r^{p-1} + \frac{a^{2p}}{r^{p+1}}\right) f(r)\, dr, \quad |\mathrm{Re}(p)| < r, \tag{8.7.5}$$

and its inverse given by

$$\mathscr{M}_+^{-1}[f(p)] = f(r) = \frac{1}{2\pi i}\int_L r^{-p}\, F_+(p) dp, \quad r > a. \tag{8.7.6}$$

In this case, we can show by integration by parts that

$$\mathscr{M}_+\left[r^2\,\frac{\partial^2 f}{\partial r^2} + r\,\frac{\partial f}{\partial r}\right] = p^2 F_+(p) - 2\, a^{p+1} f'(a), \tag{8.7.7}$$

where $f'(r)$ exists at $r = a$.

THEOREM 8.7.1 **(*Convolution*).**
If $\mathscr{M}_+\{f(r)\} = F_+(p)$, and $\mathscr{M}_+\{g(r)\} = G_+(p)$, then

$$\mathscr{M}_+\{f(r)\, g(r)\} = \frac{1}{2\pi i}\int_L F_+(\xi) G_+(p-\xi)\, d\xi. \tag{8.7.8}$$

Or, equivalently,

$$f(r) g(r) = \mathscr{M}_+^{-1}\left[\frac{1}{2\pi i}\int_L F_+(\xi)\, G_+(p-\xi) d\xi\right]. \tag{8.7.9}$$

PROOF We assume that $F_+(p)$ and $G_+(p)$ are analytic in some strip

$|\mathrm{Re}(p)| < \gamma$. Then

$$\begin{aligned}
\mathscr{M}_+\{f(r)\,g(r)\} &= \int_a^\infty \left(r^{p-1} + \frac{a^{2p}}{r^{p+1}}\right) f(r)g(r)dr \\
&= \int_a^\infty r^{p-1} f(r)g(r)dr + \int_a^\infty \frac{a^{2p}}{r^{p+1}} f(r)g(r)dr \qquad (8.7.10) \\
&= \frac{1}{2\pi i}\int_L F_+(\xi)d\xi \int_a^\infty r^{p-\xi-1} g(r)dr \\
&\qquad + \frac{1}{2\pi}\int_a^\infty \frac{a^{2p}}{r^{p+1}}\, g(r)dr \int_L r^{-\xi} F_+(\xi)\, d\xi. \qquad (8.7.11)
\end{aligned}$$

Replacing ξ by $-\xi$ in the first integral term and using $F_+(\xi) = a^{2\xi} F_+(-\xi)$, which follows from the definition (8.7.5), we obtain

$$\int_L r^{-\xi}\, F_+(\xi)d\xi = \int_L r^{\xi}\, a^{-2\xi}\, F_+(\xi)d\xi. \qquad (8.7.12)$$

The path of integration L, $\mathrm{Re}(\xi) = c$, becomes $\mathrm{Re}(\xi) = -c$, but these paths can be reconciled if $F(\xi)$ tends to zero for large $\mathrm{Im}(\xi)$.

In view of (8.7.11), we have rewritten

$$\int_a^\infty \frac{a^{2p}}{r^{p+1}} f(r)\, g(r)dr = \frac{1}{2\pi i}\int_L F_+(\xi)d\xi \int_a^\infty \frac{a^{2p-2\xi}}{r^{p-\xi+1}}\, g(r)\, dr. \qquad (8.7.13)$$

This result is used to rewrite (8.7.10) as

$$\begin{aligned}
\mathscr{M}_+\{f(r)g(r)\} &= \int_a^\infty \left(r^{p-1} + \frac{a^{2p}}{r^{p+1}}\right) f(r)g(r)dr \\
&= \int_a^\infty r^{p-1} f(r)\, g(r)\, dr + \int_a^\infty \frac{a^{2p}}{r^{p+1}} f(r)\, g(r)\, dr \\
&= \frac{1}{2\pi i}\int_L F_+(\xi)\, d\xi \int_a^\infty r^{p-\xi-1} g(r)\, dr \\
&\qquad + \frac{1}{2\pi i}\int_L F_+(\xi)d\xi \int_a^\infty \frac{a^{2p-2\xi}}{r^{p-\xi+1}}\, g(r)\, dr \\
&= \frac{1}{2\pi i}\int_L F_+(\xi)\, G_+(p-\xi)\, d\xi.
\end{aligned}$$

This completes the proof. ∎

If the range of integration is finite, then we define the *generalized finite Mellin transform* by

$$\mathscr{M}_{-}^{a}\{f(r)\} = F_{-}^{a}(p) = \int_{0}^{a} \left(r^{p-1} - \frac{a^{2p}}{r^{p+1}}\right) f(r)dr, \tag{8.7.14}$$

where $\operatorname{Re} p < \gamma$.

The corresponding inverse transform is given by

$$f(r) = -\frac{1}{2\pi i} \int_{L} \left(\frac{r}{a^2}\right)^{p} F_{-}^{a}(p)dp, \quad 0 < r < a,$$

which is, by replacing p by $-p$ and using $F_{-}^{a}(-p) = -a^{-2p} F_{-}^{a}(p)$,

$$= \frac{1}{2\pi i} \int_{L} r^{-p} F_{-}^{a}(p)dp, \quad 0 < r < a, \tag{8.7.15}$$

where the path L is Re $p = -c$ with $|c| < \gamma$.

It is easy to verify the result

$$\begin{aligned}\mathscr{M}_{-}^{a}\{r^2 f_{rr} + r f_{-r}\} &= \int_{0}^{a} \left(r^{p-1} - \frac{a^{2p}}{r^{p+1}}\right) \{r^2 f_{rr} + r f_r\} dr \\ &= p^2 F_{-}^{a}(p) - 2\, p\, a^p\, f(a).\end{aligned} \tag{8.7.16}$$

This is a useful result for applications.

Similarly, we define the generalized finite Mellin transform-pair by

$$\mathscr{M}_{+}^{a}\{f(r)\} = F_{+}^{a}(p) = \int_{0}^{a} \left(r^{p-1} + \frac{a^{2p}}{r^{p+1}}\right) f(r)\, dr, \tag{8.7.17}$$

$$f(r) = \left(\mathscr{M}_{+}^{a}\right)^{-1} \left[F_{+}^{a}(p)\right] = \frac{1}{2\pi i} \int_{L} r^{-p} F_{+}^{a}(p)\, dp, \tag{8.7.18}$$

where $|\operatorname{Re} p| < \gamma$.

For this finite transform, we can also prove

$$\begin{aligned}\mathscr{M}_{+}^{a}\left[r^2 f_{rr} + r\ f_r\right] &= \int_{0}^{a} \left(r^{p-1} + \frac{a^{2p}}{r^{p+1}}\right) \left(r^2 f_{rr} + r\ f_r\right) dr \\ &= p^2 F_{+}^{a}(p) + 2\, a^{p-1}\ f'(a).\end{aligned} \tag{8.7.19}$$

This result also seems to be useful for applications. The reader is referred to Naylor (1963) for applications of the above results to boundary value problems.

8.8 Exercises

1. Find the Mellin transform of each of the following functions:

(a) $f(x) = H(a-x)$, $a > 0$,
(b) $f(x) = x^m e^{-nx}$, $m, n > 0$,
(c) $f(x) = \frac{1}{1+x^2}$,
(d) $f(x) = J_0^2(x)$,
(e) $f(x) = x^z H(x - x_0)$,
(f) $f(x) = [H(x-x_0) - H(x)]x^z$,
(g) $f(x) = Ei(x)$,
(h) $f(x) = e^x Ei(x)$,
(i) $f(x) = \exp\left(-ax^2\right)$, $a > 0$,
(j) $f(x) = erfc(x)$,
(k) $f(x) = Ci(x)$,
(l) $f(x) = (1+x^a)^{-b}$,
(m) $f(x) = (1+x)^{-1}$,

where the exponential integral is defined by

$$Ei(x) = \int_x^\infty t^{-1}\, e^{-t}\, dt = \int_1^\infty \xi^{-1}\, e^{-\xi\, x}\, d\xi.$$

2. Derive the Mellin transform-pairs from the bilateral Laplace transform and its inverse given by

$$\bar{g}(p) = \int_{-\infty}^{\infty} e^{-pt} g(t) dt, \qquad g(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{pt}\, \bar{g}(p) dp.$$

3. Show that

$$\mathscr{M}\left[\frac{1}{e^x + e^{-x}}\right] = \Gamma(p)\, L(p),$$

where $L(p) = \dfrac{1}{1^p} - \dfrac{1}{3^p} + \dfrac{1}{5^p} - \cdots$ is the *Dirichlet L-function.*

4. Show that

$$\mathscr{M}\left\{\frac{1}{(1+ax)^n}\right\} = \frac{\Gamma(p)\Gamma(n-p)}{a^p\, \Gamma(n)}.$$

5. Show that

$$\mathscr{M}\left\{x^{-n} J_n(ax)\right\} = \frac{1}{2}\left(\frac{a}{2}\right)^{n-p} \frac{\Gamma\left(\frac{p}{2}\right)}{\Gamma\left(n - \frac{p}{2} + 1\right)}, \quad a > 0,\ n > -\frac{1}{2}.$$

6. Show that

(a) $\mathscr{M}^{-1}\left[\cos\left(\frac{\pi p}{2}\right)\Gamma(p)\,\tilde{f}(1-p)\right]=\mathscr{F}_c\left\{\sqrt{\frac{\pi}{2}}\,f(x)\right\}$,

(b) $\mathscr{M}^{-1}\left[\sin\left(\frac{\pi p}{2}\right)\Gamma(p)\,\tilde{f}(1-p)\right]=\mathscr{F}_s\left\{\sqrt{\frac{\pi}{2}}\,f(x)\right\}$.

7. If $I_n^\infty f(x)$ denotes the nth repeated integral of $f(x)$ defined by

$$I_n^\infty f(x)=\int_x^\infty I_{n-1}^\infty f(t)dt,$$

show that

(a) $\mathscr{M}\left[\int_x^\infty f(t)dt,\, p\right]=\frac{1}{p}\,\tilde{f}(p+1)$,

(b) $\mathscr{M}\left[I_n^\infty\, f(x)\right]=\frac{\Gamma(p)}{\Gamma(p+n)}\,\tilde{f}(p+n)$.

8. Show that the integral equation

$$f(x)=h(x)+\int_0^\infty g(x\xi)\ f(\xi)\ d\xi$$

has the formal solution

$$f(x)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left[\frac{\tilde{h}(p)+\tilde{g}(p)\,\tilde{h}(1-p)}{1-\tilde{g}(p)\,\tilde{g}(1-p)}\right]x^{-p}\,dp.$$

9. Find the solution of the Laplace integral equation

$$\int_0^\infty e^{-x\xi}\ f(\xi)\ d\xi=\frac{1}{(1+x)^n}.$$

10. Show that the integral equation

$$f(x)=h(x)+\int_0^\infty\ f(\xi)\ g\left(\frac{x}{\xi}\right)\ \frac{d\xi}{\xi}$$

has the formal solution

$$f(x)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{x^{-p}\,\tilde{h}(p)}{1-\tilde{g}(p)}\,dp.$$

11. Show that the solution of the integral equation

$$f(x) = e^{-ax} + \int_0^\infty \exp\left(-\frac{x}{\xi}\right) \, f(\xi) \, \frac{d\xi}{\xi}$$

is

$$f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} (ax)^{-p} \left\{ \frac{\Gamma(p)}{1 - \Gamma(p)} \right\} dp.$$

12. Assuming (see Harrington, 1967)

$$\mathscr{M}\left[f(re^{i\theta})\right] = \int_0^\infty r^{p-1} f(re^{i\theta}) \, dr, \quad p \text{ is real},$$

and putting $re^{i\theta} = \xi$, $\mathscr{M}\{f(\xi)\} = F(p)$ show that

(a) $\mathscr{M}\,[f(re^{i\theta});\, r \to p] = \exp(-ip\theta)\, F(p)$.

Hence, deduce

(b) $\mathscr{M}^{-1}\{F(p) \cos\ p\theta\} = \mathrm{Re}[f(re^{i\theta})]$,

(c) $\mathscr{M}^{-1}\{F(p) \sin\ p\theta\} = -\mathrm{Im}[f(re^{i\theta})]$.

13. (a) If $\mathscr{M}\,[\exp(-r)] = \Gamma(p)$, show that

$$\mathscr{M}\left[\exp(-re^{i\theta})\right] = \Gamma(p)\, e^{-ip\theta},$$

(b) If $\mathscr{M}\,[\log(1+r)] = \dfrac{\pi}{p \sin \pi p}$, then show that

$$\mathscr{M}\left[\mathrm{Re}\, \log(1 + re^{i\theta})\right] = \frac{\pi \cos\ p\theta}{p \sin \pi p}.$$

14. Use $\mathscr{M}^{-1}\left\{\dfrac{\pi}{\sin p\pi}\right\} = \dfrac{1}{1+x} = f(x)$, and Exercises 12(b) and 12(c), respectively, to show that

(a) $\mathscr{M}^{-1}\left\{\dfrac{\pi \cos p\theta}{\sin p\pi};\, p \to r\right\} = \dfrac{1 + r\cos\theta}{1 + 2r\cos\theta + r^2}$,

(b) $\mathscr{M}^{-1}\left\{\dfrac{\pi \sin p\theta}{\sin p\pi};\, p \to r\right\} = \dfrac{r\sin\theta}{1 + 2r\cos\theta + r^2}$.

15. Find the inverse Mellin transforms of

(a) $\Gamma(p)\cos p\theta$, where $-\dfrac{\pi}{2} < \theta < \dfrac{\pi}{2}$, (b) $\Gamma(p)\sin p\theta$.

16. Obtain the solution of Example 8.4.2 with the boundary data

    (a) $\phi(r, \alpha) = \phi(r, -\alpha) = H(a - r)$.

    (b) Solve equation (8.4.5) in $0 < r < \infty$, $0 < \theta < \alpha$ with the boundary conditions $\phi(r, 0) = 0$ and $\phi(r, \alpha) = f(r)$.

17. Show that

    (a) $\displaystyle\sum_{n=1}^{\infty} \frac{\cos kn}{n^2} = \left[\frac{k^2}{4} - \frac{\pi k}{2} + \frac{\pi^2}{6}\right]$, and (b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

18. If $f(x) = \sum_{n=1}^{\infty} a_n e^{-nx}$, show that

$$\mathscr{M}\{f(x)\} = \tilde{f}(p) = \Gamma(p)\, g(p),$$

where $g(p) = \displaystyle\sum_{n=1}^{\infty} a_n\, n^{-p}$ is the Dirichlet series.
If $a_n = 1$ for all n, derive

$$\tilde{f}(p) = \Gamma(p)\, \zeta(p).$$

Show that

$$\mathscr{M}\left\{\frac{\exp(-ax)}{1 - e^{-x}}\right\} = \Gamma(p)\, \xi(p, a).$$

19. Show that

    (a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^p} = (1 - 2^{1-p})\, \zeta(p)$.

    (b) $\displaystyle\mathscr{M}\left\{\sum_{n=1}^{\infty} (-1)^{n-1} f(nx)\right\} = (1 - 2^{1-p})\, \zeta(p) \tilde{f}(p)$.

Hence, deduce

    (c) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = \frac{\pi^2}{12}$, (d) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^4} = \left(\frac{7}{8}\right) \frac{\pi^4}{90}$.

20. Find the sum of the following series

    (a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} \cos kn$, (b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \sin kn$.

21. Show that the solution of the boundary value problem

$$r^2\phi_{rr} + r\phi_r + \phi_{\theta\theta} = 0, \quad 0 < r < \infty,\ 0 < \theta < \pi$$

$$\phi(r, 0) = \phi(r, \pi) = f(r),$$

is

$$\phi(r, \theta) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} r^{-p} \, \frac{\tilde{f}(p)\ \cos\left\{p\left(\theta - \frac{\pi}{2}\right)\right\} dp}{\cos\left(\frac{\pi p}{2}\right)}.$$

22. Evaluate

$$\sum_{n=1}^{\infty} \frac{\cos an}{n^3} = \frac{1}{12}(a^3 - 3\pi a^2 + 2\pi^2 a).$$

23. Prove the following results:

(a) $\mathscr{M}\left[\int_0^\infty \xi^n f(x\xi)\, g(\xi) d\xi\right] = \tilde{f}(p)\, \tilde{g}(1 + n - p),$

(b) $\mathscr{M}\left[\int_0^\infty \xi^n f\left(\frac{x}{\xi}\right) g(\xi) d\xi\right] = \tilde{f}(p)\, \tilde{g}(p + n + 1).$

24. Show that

(a) $W^{-\alpha}[e^{-x}] = e^{-x}, \quad \alpha > 0,$

(b) $W^{\frac{1}{2}}\left[\frac{1}{\sqrt{x}} \exp\left(-\sqrt{x}\right)\right] = \frac{K_1(\sqrt{x})}{\sqrt{\pi x}}, \quad x > 0,$

where $K_1(x)$ is the modified Bessel function of the second kind and order one.

25. (a) Show that the integral (Wong, 1989, pp. 186–187)

$$I(x) = \int_0^{\pi/2} J_\nu^2(x \cos\theta)\, d\theta, \qquad \nu > -\frac{1}{2},$$

can be written as a Mellin convolution

$$I(x) = \int_0^\infty f(x\xi)\, g(\xi)\, d\xi,$$

where

$$f(\xi) = J_\nu^2(\xi) \ \text{ and } \ \mathrm{g}(\xi) = \left\{\begin{matrix} (1 - \xi^2)^{-\frac{1}{2}}, & 0 < \xi < 1 \\ 0, & \xi \geq 1 \end{matrix}\right\}.$$

(b) Prove that the integration contour in the Parseval identity

$$I(x)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-p}\ \tilde{f}(p)\,\tilde{g}(1-p)\ dp, \qquad -2\nu<c<1,$$

cannot be shifted to the right beyond the vertical line Re $p=2$.

26. If $f(x)=\int_0^\infty \exp(-x^2t^2)\cdot\frac{\sin t}{t^2}\,J_1(t)dt$, show that

$$\mathscr{M}\{f(x)\}=\frac{\Gamma\left(p+\frac{3}{2}\right)\Gamma\left(\frac{1-p}{2}\right)}{p\,\Gamma(p+3)}.$$

27. Prove the following relations to the Laplace and the Fourier transforms:

(a) $\mathscr{M}[f(x),p]=\mathscr{L}[f(e^{-t}),p]$,
(b) $\mathscr{M}[f(x);a+i\omega]=\mathscr{F}[f(e^{-t})e^{-at};\omega]$,

where $\mathscr{L}$ is the two-sided Laplace transform and $\mathscr{F}$ is the Fourier transform without the factor $(2\pi)^{-\frac{1}{2}}$.

28. Prove the following properties of convolution:

(a) $f*g=g*f$,
(b) $(f*g)*h=f*(g*h)$,
(c) $f(x)*\delta(x-1)=f(x)$,
(d) $\delta(x-a)*f(x)=a^{-1}f\left(\frac{x}{a}\right)$,
(e) $\delta^n(n-1)*f(x)=\left(\frac{d}{dx}\right)^n(x^n f(x))$,
(f) $\left(x\frac{d}{dx}\right)^n(f*g)=\left[\left(x\frac{d}{dx}\right)^n f\right]*g=f*\left[\left(x\frac{d}{dx}\right)^n g\right]$.

29. If $\mathscr{M}\{f(r,\theta)\}=\tilde{f}(p,\theta)$ and $\nabla^2 f(r,\theta)=f_{rr}+\frac{1}{r}f_r+\frac{1}{r^2}f_{\theta\theta}$, show that

$$\mathscr{M}\left\{\nabla^2 f(r,\theta)\right\}=\left[\frac{d^2}{d\theta^2}+(p-2)^2\right]\tilde{f}(p-2,\theta).$$