CHAPTER 28

Series Solutions Near a Regular Singular Point

REGULAR SINGULAR POINTS

The point x_0 is a *regular singular point* of the second-order homogeneous linear differential equation

$$y'' + P(x)y' + Q(x)y = 0 \tag{28.1}$$

if x_0 is not an ordinary point (see Chapter 27) but both $(x - x_0)P(x)$ and $(x - x_0)^2Q(x)$ are analytic at x_0. We only consider regular singular points at $x_0 = 0$; if this is not the case, then the change of variables $t = x - x_0$ will translate x_0 to the origin.

METHOD OF FROBENIUS

Theorem 28.1. If $x = 0$ is a regular singular point of *(28.1)*, then the equation has at least one solution of the form

$$y = x^{\lambda} \sum_{n=0}^{\infty} a_n x^n$$

where λ and a_n ($n = 0, 1, 2, \ldots$) are constants. This solution is valid in an interval $0 < x < R$ for some real number R.

To evaluate the coefficients a_n and λ in Theorem 28.1, one proceeds as in the power series method of Chapter 27. The infinite series

$$\begin{aligned} y &= x^{\lambda} \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{\lambda+n} \\ &= a_0 x^{\lambda} + a_1 x^{\lambda+1} + a_2 x^{\lambda+2} + \cdots + a_{n-1} x^{\lambda+n-1} + a_n x^{\lambda+n} + a_{n+1} x^{\lambda+n+1} + \cdots \end{aligned} \tag{28.2}$$

with its derivatives

$$\begin{aligned} y' &= \lambda a_0 x^{\lambda-1} + (\lambda+1)a_1 x^{\lambda} + (\lambda+2)a_2 x^{\lambda+1} + \cdots \\ &\quad + (\lambda+n-1)a_{n-1} x^{\lambda+n-2} + (\lambda+n)a_n x^{\lambda+n-1} + (\lambda+n+1)a_{n+1} x^{\lambda+n} + \cdots \end{aligned} \tag{28.3}$$

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and
$$y'' = \lambda(\lambda-1)a_0x^{\lambda-2} + (\lambda+1)(\lambda)a_1x^{\lambda-1} + (\lambda+2)(\lambda+1)a_2x^{\lambda} + \cdots$$
$$+(\lambda+n-1)(\lambda+n-2)a_{n-1}x^{\lambda+n-3} + (\lambda+n)(\lambda+n-1)a_nx^{\lambda+n-2}$$
$$+(\lambda+n+1)(\lambda+n)a_{n+1}x^{\lambda+n-1} + \cdots \tag{28.4}$$

are substituted into Eq. (*28.1*). Terms with like powers of x are collected together and set equal to zero. When this is done for x^n the resulting equation is a recurrence formula. A quadratic equation in λ, called the *indicial equation*, arises when the coefficient of x^0 is set to zero and a_0 is left arbitrary.

The two roots of the indicial equation can be real or complex. If complex they will occur in a conjugate pair and the complex solutions that they produce can be combined (by using Euler's relations and the identity $x^{a\pm ib} = x^a e^{\pm ib \ln x}$) to form real solutions. In this book we shall, for simplicity, suppose that both roots of the indicial equation are real. Then, if λ is taken as the *larger* indicial root, $\lambda = \lambda_1 \geq \lambda_2$, the method of Frobenius always yields a solution

$$y_1(x) = x^{\lambda_1}\sum_{n=0}^{\infty} a_n(\lambda_1)x^n \tag{28.5}$$

to Eq. (*28.1*). [We have written $a_n(\lambda_1)$ to indicate the coefficients produced by the method when $\lambda = \lambda_1$.]

If $P(x)$ and $Q(x)$ are quotients of polynomials, it is usually easier first to multiply (*28.1*) by their lowest common denominator and then to apply the method of Frobenius to the resulting equation.

GENERAL SOLUTION

The method of Frobenius always yields one solution to (*28.1*) of the form (*28.5*). The general solution (see Theorem 8.2) has the form $y = c_1y_1(x) + c_2y_2(x)$ where c_1 and c_2 are arbitrary constants and $y_2(x)$ is a second solution of (*28.1*) that is linearly independent from $y_1(x)$. The method for obtaining this second solution depends on the relationship between the two roots of the indicial equation.

Case 1. If $\lambda_1 - \lambda_2$ is not an integer, then

$$y_2(x) = x^{\lambda_2}\sum_{n=0}^{\infty} a_n(\lambda_2)x^n \tag{28.6}$$

where $y_2(x)$ is obtained in an identical manner as $y_1(x)$ by the method of Frobenius, using λ_2 in place of λ_1.

Case 2. If $\lambda_1 = \lambda_2$, then

$$y_2(x) = y_1(x)\ln x + x^{\lambda_1}\sum_{n=0}^{\infty} b_n(\lambda_1)x^n \tag{28.7}$$

To generate this solution, keep the recurrence formula in terms of λ and use it to find the coefficients a_n $(n \geq 1)$ in terms of both λ and a_0, where the coefficient a_0 remains arbitrary. Substitute these a_n into Eq. (*28.2*) to obtain a function $y(\lambda, x)$ which depends on the variables λ and x. Then

$$y_2(x) = \left.\frac{\partial y(\lambda, x)}{\partial \lambda}\right|_{\lambda=\lambda_1} \tag{28.8}$$

Case 3. If $\lambda_1 - \lambda_2 = N$, a positive integer, then

$$y_2(x) = d_{-1}y_1(x)\ln x + x^{\lambda_2}\sum_{n=0}^{\infty} d_n(\lambda_2)x^n \tag{28.9}$$

To generate this solution, first try the method of Frobenius. with λ_2. If it yields a second solution, then this solution is $y_2(x)$, having the form of (*28.9*) with $d_{-1} = 0$. Otherwise, proceed as in Case 2 to generate $y(\lambda, x)$, whence

$$y_2(x) = \frac{\partial}{\partial \lambda}[(\lambda - \lambda_2)y(\lambda, x)]\Big|_{\lambda=\lambda_2} \tag{28.10}$$

Solved Problems

28.1. Determine whether $x = 0$ is a regular singular point of the differential equation

$$y'' - xy' + 2y = 0$$

As shown in Problem 27.1, $x = 0$ is an ordinary pont of this differential equation, so it cannot be a regular singular point.

28.2. Determine whether $x = 0$ is a regular singular point of the differential equation

$$2x^2y'' + 7x(x+1)y' - 3y = 0$$

Dividing by $2x^2$, we have

$$P(x) = \frac{7(x+1)}{2x} \quad \text{and} \quad Q(x) = \frac{-3}{2x^2}$$

As shown in Problem 27.7, $x = 0$ is a singular point. Furthermore, both

$$xP(x) = \frac{7}{2}(x+1) \quad \text{and} \quad x^2Q(x) = -\frac{3}{2}$$

are analytic everywhere: the first is a polynomial and the second a constant. Hence, both are analytic at $x = 0$, and this point is a regular singular point.

28.3. Determine whether $x = 0$ is a regular singular point of the differential equation

$$x^3y'' + 2x^2y' + y = 0$$

Dividing by x^3, we have

$$P(x) = \frac{2}{x} \quad \text{and} \quad Q(x) = \frac{1}{x^3}$$

Neither of these functions is defined at $x = 0$, so this point is a singular point. Here,

$$xP(x) = 2 \quad \text{and} \quad x^2Q(x) = \frac{1}{x}$$

The first of these terms is analytic everywhere, but the second is undefined at $x = 0$ and not analytic there. Therefore, $x = 0$ is *not* a regular singular point for the given differential equation.

28.4. Determine whether $x = 0$ is a regular singular point of the differential equation

$$8x^2y'' + 10xy' + (x-1)y = 0$$

Dividing by $8x^2$, we have

$$P(x) = \frac{5}{4x} \quad \text{and} \quad Q(x) = \frac{1}{8x} - \frac{1}{8x^2}$$

Neither of these functions is defined at $x = 0$, so this point is a singular point. Furthermore, both

$$xP(x) = \frac{5}{4} \quad \text{and} \quad x^2Q(x) = \frac{1}{8}(x-1)$$

are analytic everywhere: the first is a constant and the second a polynomial. Hence, both are analytic at $x = 0$, and this point is a regular singular point.

28.5. **Find a recurrence formula and the indicial equation for an infinite series solution around $x = 0$ for the differential equation given in Problem 28.4.**

It follows from Problem 28.4 that $x = 0$ is a regular singular point of the differential equation, so Theorem 24.1 holds. Substituting Eqs. (*28.2*) through (*28.4*) into the left side of the given differential equation and combining coefficients of like powers of x, we obtain

$$x^{\lambda}[8\lambda(\lambda-1)a_0 + 10\lambda a_0 - a_0] + x^{\lambda+1}[8(\lambda+1)\lambda a_1 + 10(\lambda+1)a_1 + a_0 - a_1] + \cdots$$
$$+ x^{\lambda+n}[8(\lambda+n)(\lambda+n-1)a_n + 10(\lambda+n)a_n + a_{n-1} - a_n] + \cdots = 0$$

Dividing by x^{λ} and simplifying, we have

$$[8\lambda^2 + 2\lambda - 1]a_0 + x[(8\lambda^2 + 18\lambda + 9)a_1 + a_0] + \cdots$$
$$+ x^n\{[8(\lambda+n)^2 + 2(\lambda+n) - 1]a_n + a_{n-1}\} + \cdots = 0$$

Factoring the coefficient of a_n and equating the coefficient of each power of x to zero, we find

$$(8\lambda^2 + 2\lambda - 1)a_0 = 0 \tag{1}$$

and, for $n \geq 1$,

$$[4(\lambda+n) - 1][2(\lambda+n) + 1]a_n + a_{n-1} = 0$$

or,

$$a_n = \frac{-1}{[4(\lambda+n)-1][2(\lambda+n)+1]}a_{n-1} \tag{2}$$

Equation (*2*) is a recurrence formula for this differential equation.

From (*1*), either $a_0 = 0$ or

$$8\lambda^2 + 2\lambda - 1 = 0 \tag{3}$$

It is convenient to keep a_0 arbitrary; therefore, we must choose λ to satisfy (3), which is the indicial equation.

28.6. **Find the general solution near $x = 0$ of $8x^2y'' + 10xy' + (x - 1)y = 0$.**

The roots of the indicial equation given by (*3*) of Problem 28.5 are $\lambda_1 = \frac{1}{4}$, and $\lambda_2 = -\frac{1}{2}$. Since $\lambda_1 - \lambda_2 = \frac{3}{4}$, the solution is given by Eqs. (*28.5*) and (*28.6*). Substituting $\lambda = \frac{1}{4}$ into the recurrence formula (*2*) of Problem 28.5 and simplifying, we obtain

$$a_n = \frac{-1}{2n(4n+3)}a_{n-1} \quad (n \geq 1)$$

Thus,

$$a_1 = \frac{-1}{14}a_0, \quad a_2 = \frac{-1}{44}a_1 = \frac{1}{616}a_0, \quad \ldots$$

and

$$y_1(x) = a_0x^{1/4}\left(1 - \frac{1}{14}x + \frac{1}{616}x^2 + \cdots\right)$$

Substituting $\lambda = -\frac{1}{2}$ into recurrence formula (*2*) of Problem 28.5 and simplifying, we obtain

$$a_n = \frac{-1}{2n(4n-3)}a_{n-1}$$

Thus,

$$a_1 = -\frac{1}{2}a_0, \quad a_2 = \frac{-1}{20}a_1 = \frac{1}{40}a_0, \quad \ldots$$

and

$$y_2(x) = a_0x^{-1/2}\left(1 - \frac{1}{2}x + \frac{1}{40}x^2 + \cdots\right)$$

The general solution is

$$y = c_1y_1(x) + c_2y_2(x)$$
$$= k_1x^{1/4}\left(1 - \frac{1}{14}x + \frac{1}{616}x^2 + \cdots\right) + k_2x^{-1/2}\left(1 - \frac{1}{2}x + \frac{1}{40}x^2 + \cdots\right)$$

where $k_1 = c_1a_0$ and $k_2 = c_2a_0$.

28.7. Find a recurrence formula and the indicial equation for an infinite series solution around $x = 0$ for the differential equation

$$2x^2y'' + 7x(x + 1)y' - 3y = 0$$

It follows from Problem 28.2 that $x = 0$ is a regular singular point of the differential equation, so Theorem 28.1 holds. Substituting Eqs. (*28.2*) through (*28.4*) into the left side of the given differential equation and combining coefficients of like powers of x, we obtain

$$x^\lambda[2\lambda(\lambda - 1)a_0 + 7\lambda a_0 - 3a_0] + x^{\lambda+1}[2(\lambda + 1)\lambda a_1 + 7\lambda a_0 + 7(\lambda + 1)a_1 - 3a_1] + \cdots$$
$$+ x^{\lambda+n}[2(\lambda + n)(\lambda + n - 1)a_n + 7(\lambda + n - 1)a_{n-1} + 7(\lambda + n)a_n - 3a_n] + \cdots 0$$

Dividing by x^λ and simplifying, we have

$$(2\lambda^2 + 5\lambda - 3)a_0 + x[(2\lambda^2 + 9\lambda + 4)a_1 + 7\lambda a_0] + \cdots$$
$$+ x^n\{[2(\lambda + n)^2 + 5(\lambda + n) - 3]a_n + 7(\lambda + n - 1)a_{n-1}\} + \cdots = 0$$

Factoring the coefficient of a_n and equating each coefficient to zero, we find

$$(2\lambda^2 + 5\lambda - 3)a_0 = 0 \tag{1}$$

and, for $n \geq 1$,

$$[2(\lambda + n) - 1][(\lambda + n) + 3]a_n + 7(\lambda + n - 1)a_{n-1} = 0$$

or,

$$a_n = \frac{-7(\lambda + n - 1)}{[2(\lambda + n) - 1][(\lambda + n) + 3]}a_{n-1} \tag{2}$$

Equation (*2*) is a recurrence formula for this differential equation.

From (*1*), either $a_0 = 0$ or

$$2\lambda^2 + 5\lambda - 3 = 0 \tag{3}$$

It is convenient to keep a_0 arbitrary; therefore, We require λ to satisfy the indicial equation (*3*).

28.8. Find the general solution near $x = 0$ of $2x^2y'' + 7x(x + 1)y' - 3y = 0$.

The roots of the indicial equation given by (*3*) of Problem 28.7 are $\lambda_1 = \frac{1}{2}$ and $\lambda_2 = -3$. Since $\lambda_1 - \lambda_2 = \frac{7}{2}$, the solution is given by Eqs. (*28.5*) and (*28.6*). Substituting $\lambda = \frac{1}{2}$ into (*2*) of Problem 28.7 and simplifying, we obtain

$$a_n = \frac{-7(2n - 1)}{2n(2n + 7)}a_{n-1} \quad (n \geq 1)$$

Thus,

$$a_1 = -\frac{7}{18}a_0, \quad a_2 = -\frac{21}{44}a_1 = \frac{147}{792}a_0, \quad \ldots$$

and

$$y_1(x) = a_0x^{1/2}\left(1 - \frac{7}{18}x + \frac{147}{792}x^2 + \cdots\right)$$

Substituting $\lambda = -3$ into (*2*) of Problem 28.7 and simplifying, we obtain

$$a_n = \frac{-7(n-4)}{n(2n-7)} a_{n-1} \quad (n \geq 1)$$

Thus, $$a_1 = -\frac{21}{5}a_0, \quad a_2 = -\frac{7}{3}a_1 = \frac{49}{5}a_0, \quad a_3 = -\frac{7}{3}a_2 = -\frac{343}{15}a_0, \quad a_4 = 0$$

and, since $a_4 = 0$, $a_n = 0$ for $n \geq 4$. Thus,

$$y_2(x) = a_0 x^{-3}\left(1 - \frac{21}{5}x + \frac{49}{5}x^2 - \frac{343}{15}x^3\right)$$

The general solution is

$$\begin{aligned} y &= c_1 y_1(x) + c_2 y_2(x) \\ &= k_1 x^{1/2}\left(1 - \frac{7}{18}x + \frac{147}{792}x^2 + \cdots\right) + k_2 x^{-3}\left(1 - \frac{21}{5}x + \frac{49}{5}x^2 - \frac{343}{15}x^3\right) \end{aligned}$$

where $k_1 = c_1 a_0$ and $k_2 = c_2 a_0$.

28.9. Find the general solution near $x = 0$ of $3x^2 y'' - xy' + y = 0$.

Here $P(x) = -1/(3x)$ and $Q(x) = 1/(3x^2)$; hence, $x = 0$ is a regular singular point and the method of Frobenius is applicable. Substituting Eqs. (*28.2*) through (*28.4*) into the differential equation and simplifying, we have

$$x^\lambda[3\lambda^2 - 4\lambda + 1]a_0 + x^{\lambda+1}[3\lambda^2 + 2\lambda]a_1 + \cdots + x^{\lambda+n}[3(\lambda + \text{n})^2 - 4(\lambda + n) + 1]a_n + \cdots = 0$$

Dividing by x^λ and equating all coefficients to zero, we find

$$(3\lambda^2 - 4\lambda + 1)a_0 = 0 \qquad (1)$$

and $$[3(\lambda + n)^2 - 4(\lambda + n) + 1]a_n = 0 \quad (n \geq 1) \qquad (2)$$

From (*1*), we conclude that the indicial equation is $3\lambda^2 - 4\lambda + 1 = 0$, which has roots $\lambda_1 = 1$ and $\lambda_2 = \frac{1}{3}$.

Since $\lambda_1 - \lambda_2 = \frac{2}{3}$, the solution is given by Eqs. (*28.5*) and (*28.6*). Note that for either value of λ, (*2*) is satisfied by simply choosing $a_n = 0$, $n \geq 1$. Thus,

$$y_1(x) = x^1 \sum_{n=0}^{\infty} a_n x^n = a_0 x \qquad y_2(x) = x^{1/3} \sum_{n=0}^{\infty} a_n x^n = a_0 x^{1/3}$$

and the general solution is

$$y = c_1 y_1(x) + c_2 y_2(x) = k_1 x + k_2 x^{1/3}$$

where $k_1 = c_1 a_0$ and $k_2 = c_2 a_0$.

28.10. Use the method of Frobenius to find one solution near $x = 0$ of $x^2 y'' + xy' + x^2 y = 0$.

Here $P(x) = 1/x$ and $Q(x) = 1$, so $x = 0$ is a regular singular point and the method of Frobenius is applicable. Substituting Eqs. (*28.2*) through (*28.4*) into the left side of the differential equation, as given, and combining coefficients of like powers of x, we obtain

$$x^\lambda[\lambda^2 a_0] + x^{\lambda+1}[(\lambda + 1)^2 a_1] + x^{\lambda+2}[(\lambda + 2)^2 a_2 + a_0] + \cdots + x^{\lambda+n}[(\lambda + n)^2 a_n + a_{n-2}] + \cdots = 0$$

Thus, $$\lambda^2 a_0 = 0 \qquad (1)$$

$$(\lambda + 1)^2 a_1 = 0 \qquad (2)$$

and, for $n \geq 2$, $(\lambda + n)^2 a_n + a_{n-2} = 0$, or,

$$a_n = \frac{-1}{(\lambda + n)^2} a_{n-2} \quad (n \geq 2) \qquad (3)$$

The stipulation $n \geq 2$ is required in (*3*) because a_{n-2} is not defined for $n = 0$ or $n = 1$. From (*1*), the indicial equation is $\lambda^2 = 0$, which has roots, $\lambda_1 = \lambda_2 = 0$. Thus, we will obtain only *one* solution of the form of (*28.5*); the second solution, $y_2(x)$, will have the form of (*28.7*).

Substituting $\lambda = 0$ into (*2*) and (*3*), we find that $a_1 = 0$ and $a_n = -(1/n^2)a_{n-2}$. Since $a_1 = 0$, it follows that $0 = a_3 = a_5 = a_7 = \cdots$. Furthermore,

$$a_2 = -\frac{1}{4}a_0 = -\frac{1}{2^2(1!)^2}a_0 \qquad a_4 = -\frac{1}{16}a_2 = -\frac{1}{2^4(2!)^2}a_0$$

$$a_6 = -\frac{1}{36}a_4 = -\frac{1}{2^6(3!)^2}a_0 \qquad a_8 = -\frac{1}{64}a_6 = \frac{1}{2^8(4!)^2}a_0$$

and, in general, $a_{2k} = \dfrac{(-1)k}{2^{2k}(k!)^2}a_0$ $(k = 1, 2, 3, \ldots)$. Thus,

$$y_1(x) = a_0x^0\left[1 - \frac{1}{2^2(1!)^2}x^2 + \frac{1}{2^4(2!)^2}x^4 + \cdots + \frac{(-1)^k}{2^{2k}(k!)^2}x^{2k} + \cdots\right]$$

$$= a_0\sum_{n=0}^{\infty}\frac{(-1)^n}{2^{2n}(n!)^2}x^{2n} \tag{4}$$

28.11. Find the general solution near $x = 0$ to the differential equation given in Problem 28.10.

One solution is given by (*4*) in Problem 28.10. Because the roots of the indicial equation are equal, we use Eq. (*28.8*) to generate a second linearly independent solution. The recurrence formula is (*3*) of Problem 28.10, augmented by (*2*) of Problem 28.10 for the special case $n = 1$. From (*2*), $a_1 = 0$, which implies that $0 = a_3 = a_5 = a_7 = \cdots$. Then, from (*3*),

$$a_2 = \frac{-1}{(\lambda+2)^2}a_0, \quad a_4 = \frac{-1}{(\lambda+4)^2}a_2 = \frac{1}{(\lambda+4)^2(\lambda+2)^2}a_0, \quad \ldots$$

Substituting these values into Eq. (*28.2*), we have

$$y(\lambda, x) = a_0\left[x^\lambda - \frac{1}{(\lambda+2)^2}x^{\lambda+2} + \frac{1}{(\lambda+4)^2(\lambda+2)^2}x^{\lambda+4} + \cdots\right]$$

Recall that $\dfrac{\partial}{\partial\lambda}(x^{\lambda+k}) = x^{\lambda+k}\ln x$. (When differentiating with respect to λ, x can be thought of as a constant.) Thus,

$$\frac{\partial y(\lambda,x)}{\partial\lambda} = a_0\left[x^\lambda \ln x + \frac{2}{(\lambda+2)^3}x^{\lambda+2} - \frac{1}{(\lambda+2)^2}x^{\lambda+2}\ln x - \frac{2}{(\lambda+4)^3(\lambda+2)^2}x^{\lambda+4} - \frac{2}{(\lambda+4)^2(\lambda+2)^3}x^{\lambda+4} + \frac{1}{(\lambda+4)^2(\lambda+2)^2}x^{\lambda+4}\ln x + \cdots\right]$$

and

$$\begin{aligned}
y_2(x) = \left.\frac{\partial y(\lambda,x)}{\partial\lambda}\right|_{\lambda=0} &= a_0\left(\ln x + \frac{2}{2^3}x^2 - \frac{1}{2^2}x^2\ln x - \frac{2}{4^3 2^2}x^4 - \frac{2}{4^2 2^3}x^4 + \frac{1}{4^2 2^2}x^4\ln x + \cdots\right) \\
&= (\ln x)a_0\left[1 - \frac{1}{2^2(1!)}x^2 + \frac{1}{2^4(2!)^2}x^4 + \cdots\right] \\
&\quad + a_0\left[\frac{x^2}{2^2(1!)}(1) - \frac{x^4}{2^4(2!)^2}\left(\frac{1}{2}+1\right) + \cdots\right] \\
&= y_1(x)\ln x + a_0\left[\frac{x^2}{2^2(1!)^2}(1) - \frac{x^4}{2^4(2!)^2}\left(\frac{3}{2}\right) + \cdots\right]
\end{aligned} \tag{1}$$

which is the form claimed in Eq. (*28.7*). The general solution is $y = c_1y_1(x) + c_2y_2(x)$.

28.12. Use the method of Frobenius to find one solution near $x = 0$ of $x^2y'' - xy' + y = 0$.

Here $P(x) = -1/x$ and $Q(x) = 1/x^2$, so $x = 0$ is a regular singular point and the method of Frobenius is applicable. Substituting Eqs. (*28.2*) through (*28.4*) into the left side of the differential equation, as given, and combining coefficients of like powers of x, we obtain

$$x^{\lambda}(\lambda - 1)^2 a_0 + x^{\lambda+1}[\lambda^2 a_1] + \cdots + x^{\lambda+n}[(\lambda+n)^2 - 2(\lambda+n) + 1]a_n + \cdots = 0$$

Thus,
$$(\lambda - 1)^2 a_0 = 0 \tag{1}$$

and, in general,
$$[(\lambda+n)^2 - 2(\lambda+n) + 1]a_n = 0 \tag{2}$$

From (*1*), the indicial equation is $(\lambda - 1)^2 = 0$, which has roots $\lambda_1 = \lambda_2 = 1$. Substituting $\lambda = 1$ into (*2*), we obtain $n^2 a_n = 0$, which implies that $a_n = 0$, $n \geq 1$. Thus, $y_1(x) = a_0 x$.

28.13. Find the general solution near $x = 0$ to the differential equation given in Problem 28.12.

One solution is given in Problem 28.12. Because the roots of the indicial equation are equal, we use Eq. (*28.8*) to generate a second linearly independent solution. The recurrence formula is (*2*) of Problem 28.12. Solving it for a_n, in terms of λ, we find that $a_n = 0$ $(n \geq 1)$, and when these values are substituted into Eq. (*28.2*), we have $y(\lambda, x) = a_0 x^{\lambda}$. Thus,

$$\frac{\partial y(\lambda, x)}{\partial \lambda} = a_0 x^{\lambda} \ln x$$

and
$$y_2(x) = \left.\frac{\partial y(\lambda, x)}{\partial \lambda}\right|_{\lambda=1} = a_0 x \ln x = y_1(x) \ln x$$

which is precisely the form of Eq. (*28.7*), where, for this particular differential equation, $b_n(\lambda_1) = 0$ $(n = 0, 1, 2, \ldots)$. The general solution is

$$y = c_1 y_1(x) + c_2 y_2(x) = k_1(x) + k_2 x \ln x$$

where $k_1 = c_1 a_0$, and $k_2 = c_2 a_0$.

28.14. Use the method of Frobenius to find one solution near $x = 0$ of $x^2y'' + (x^2 - 2x)y' + 2y = 0$.

Here
$$P(x) = 1 - \frac{2}{x} \quad \text{and} \quad Q(x) = \frac{2}{x^2}$$

so $x = 0$ is a regular singular point and the method of Frobenius is applicable. Substituting, Eqs. (*28.2*) through (*28.4*) into the left side of the differential equation, as given, and combining coefficients of like powers of x, we obtain

$$x^{\lambda}[(\lambda^2 - 3\lambda + 2)a_0] + x^{\lambda+1}[(\lambda^2 - \lambda)a_1 + \lambda a_0] + \cdots$$
$$+ x^{\lambda+n}\{[(\lambda+n)^2 - 3(\lambda+n) + 2]a_n + (\lambda + n - 1)a_{n-1}\} + \cdots = 0$$

Dividing by x^{λ}, factoring the coefficient of a_n, and equating the coefficient of each power of x to zero, we obtain

$$(\lambda^2 - 3\lambda + 2)a_0 = 0 \tag{1}$$

and, in general, $[(\lambda + n) - 2][(\lambda + n) - 1]a_n + (\lambda + n - 1)a_{n-1} = 0$, or,

$$a_n = -\frac{1}{\lambda + n - 2} a_{n-1} \quad (n \geq 1) \tag{2}$$

From (*1*), the indicial equation is $\lambda^2 - 3\lambda + 2 = 0$, which has roots $\lambda_1 = 2$ and $\lambda_2 = 1$. Since $\lambda_1 - \lambda_2 = 1$, a positive integer, the solution is given by Eqs. (*28.5*) and (*28.9*). Substituting $\lambda = 2$ into (*2*), we have $a_n = -(1/n)a_{n-1}$, from which we obtain

$$a_1 = -a_0$$
$$a_2 = -\frac{1}{2}a_1 = \frac{1}{2!}a_0$$
$$a_3 = -\frac{1}{3}a_2 = -\frac{1}{3}\frac{1}{2!}a_0 = -\frac{1}{3!}a_0$$

and, in general, $a_k = \dfrac{(-1)^k}{k!}a_0$. Thus,

$$y_1(x) = a_0x^2\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}x^n = a_0x^2e^{-x} \tag{3}$$

28.15. Find the general solution near $x = 0$ to the differential equation given in Problem 28.14.

One solution is given by (3) in Problem 28.14 for the indicial. root $\lambda_1 = 2$. If we try the method of Frobenius with the indicial root $\lambda_2 = 1$, recurrence formula (2) of Problem 28.14 becomes

$$a_n = -\frac{1}{n-1}a_{n-1}$$

which leaves a_1, undefined because the denominator is zero when $n = 1$. Instead, we must use (*28.10*) to generate a second linearly independent solution. Using the recurrence formula (2) of Problem 28.14 to solve sequentially for a_n $(n = 1, 2, 3, \ldots)$ in terms of λ, we find

$$a_1 = -\frac{1}{\lambda-1}a_0,\quad a_2 = -\frac{1}{\lambda}a_1 = \frac{1}{\lambda(\lambda-1)}a_0,\quad a_3 = -\frac{1}{\lambda+1}a_2 = \frac{-1}{(\lambda+1)\lambda(\lambda-1)}a_0,\quad \ldots$$

Substituting these values into Eq. (*28.2*) we obtain

$$y(\lambda,x) = a_0\left[x^\lambda - \frac{1}{(\lambda-1)}x^{\lambda+1} + \frac{1}{\lambda(\lambda-1)}x^{\lambda+2} - \frac{1}{(\lambda+1)\lambda(\lambda-1)}x^{\lambda+3} + \cdots\right]$$

and, since $\lambda - \lambda_2 = \lambda - 1$,

$$(\lambda-\lambda_2)y(\lambda,x) = a_0\left[(\lambda-1)x^\lambda - x^{\lambda+1} + \frac{1}{\lambda}x^{\lambda+2} - \frac{1}{\lambda(\lambda+1)}x^{\lambda+3} + \cdots\right]$$

Then

$$\frac{\partial}{\partial\lambda}[(\lambda-\lambda_2)y(\lambda,x)] = a_0\left[x^\lambda + (\lambda-1)x^\lambda\ln x - x^{\lambda+1}\ln x - \frac{1}{\lambda^2}x^{\lambda+2} + \frac{1}{\lambda}x^{\lambda+2}\ln x\right.$$
$$\left. + \frac{1}{\lambda^2(\lambda+1)}x^{\lambda+3} + \frac{1}{\lambda(\lambda+1)^2}x^{\lambda+3} - \frac{1}{\lambda(\lambda+1)}x^{\lambda+3}\ln x + \cdots\right]$$

and

$$y_2(x) = \frac{\partial}{\partial\lambda}[(\lambda-\lambda_2)y(\lambda,x)]\Big|_{\lambda=\lambda_2=1}$$
$$= a_0\left(x + 0 - x^2\ln x - x^3 + x^3\ln x + \frac{1}{2}x^4 + \frac{1}{4}x^4 - \frac{1}{2}x^4\ln x + \cdots\right)$$
$$= (-\ln x)a_0\left(x^2 - x^3 + \frac{1}{2}x^4 + \cdots\right) + a_0\left(x - x^3 + \frac{3}{4}x^4 + \cdots\right)$$
$$= -y_1(x)\ln x + a_0x\left(1 - x^2 + \frac{3}{4}x^3 + \cdots\right)$$

This is the form claimed in Eq. (*28.9*), with $d_{-1} = -1$, $d_0 = a_0$, $d_1 = 0$, $d_3 = \frac{3}{4}a_0, \ldots$. The general solution is $y = c_1y_1(x) + c_2y_2(x)$.

28.16. **Use the method of Frobenius to find one solution near $x = 0$ of $x^2y'' + xy' + (x^2 - 1)y = 0$.**

Here

$$P(x) = \frac{1}{x} \quad \text{and} \quad Q(x) = 1 - \frac{1}{x^2}$$

so $x = 0$ is a regular singular point and the method of Frobenius is applicable. Substituting Eqs. (*28.2*) through (*28.4*) into the left side of the differential equation, as given, and combining coefficients of like powers of x, we obtain

$$x^{\lambda}[(\lambda^2 - 1)a_0] + x^{\lambda+1}[(\lambda+1)^2 - 1]a_1 + x^{\lambda+2}\{[(\lambda+2)^2 - 1]a_2 + a_0\} + \cdots$$
$$+ x^{\lambda+n}\{[(\lambda+n)^2 - 1]a_n + a_{n-2}\} + \cdots = 0$$

Thus,
$$(\lambda^2 - 1)a_0 = 0 \tag{1}$$
$$[(\lambda+1)^2 - 1]a_1 = 0 \tag{2}$$

and, for $n \ge 2$, $[(\lambda+n)^2 - 1]a_n + a_{n-2} = 0$, or,

$$a_n = \frac{-1}{(\lambda+n)^2 - 1}a_{n-2} \quad (n \ge 2) \tag{3}$$

From (*1*), the indicial equation is $\lambda^2 - 1 = 0$, which has roots $\lambda_1 = 1$ and $\lambda_2 = -1$. Since $\lambda_1 - \lambda_2 = 2$, a positive integer, the solution is given by (*28.5*) and (*28.9*). Substituting $\lambda = 1$ into (*2*) and (*3*), we obtain $a_1 = 0$ and

$$a_n = \frac{-1}{n(n+2)}a_{n-2} \quad (n \ge 2)$$

Since $a_1 = 0$, it follows that $0 = a_3 = a_5 = a_7 = \cdots$. Furthermore,

$$a_2 = \frac{-1}{2(4)}a_0 = \frac{-1}{2^2 1!2!}a_0, \quad a_4 = \frac{-1}{4(6)}a_2 = \frac{1}{2^4 2!3!}a_0, \quad a_6 = \frac{-1}{6(8)}a_4 = \frac{-1}{2^6 3!4!}a_0$$

and, in general,

$$a_{2k} = \frac{(-1)^k}{2^{2k}k!(k+1)!}a_0 \quad (k = 1, 2, 3, \ldots)$$

Thus,
$$y_1(x) = a_0 x \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n} n!\,(n+1)!} x^{2n} \tag{4}$$

28.17. **Find the general solution near $x = 0$ to the differential equation given in Problem 28.16.**

One solution is given by (*4*) in Problem 28.16 for the indicial root $\lambda_1 = 1$. If we try the method of Frobenius with the indicial root $\lambda_2 = -1$, recurrence formula (*3*) of Problem 28.16 becomes

$$a_n = -\frac{1}{n(n-2)}a_{n-2}$$

which fails to define a_2 because the denominator is zero when $n = 2$. Instead, we must use Eq. (*28.10*) to generate a second linearly independent solution. Using Eqs. (*2*) and (*3*) of Problem 28.16 to solve sequentially for $a_n (n = 1, 2, 3, \ldots)$ in terms of λ, we find $0 = a_1 = a_3 = a_5 = \cdots$ and

$$a_2 = \frac{-1}{(\lambda+3)(\lambda+1)}a_0, \quad a_4 = \frac{1}{(\lambda+5)(\lambda+3)^2(\lambda+1)}a_0, \quad \ldots$$

Thus,
$$y(\lambda, x) = a_0\left[x^{\lambda} - \frac{1}{(\lambda+3)(\lambda+1)}x^{\lambda+2} + \frac{1}{(\lambda+5)(\lambda+3)^2(\lambda+1)}x^{\lambda+4} + \cdots\right]$$

Since $\lambda - \lambda_2 = \lambda + 1$,

$$(\lambda - \lambda_2)y(\lambda, x) = a_0\left[(\lambda+1)x^{\lambda} - \frac{-1}{(\lambda+3)}x^{\lambda+2} + \frac{1}{(\lambda+5)(\lambda+3)^2}x^{\lambda+4} + \cdots\right]$$

and

$$\frac{\partial}{\partial\lambda}[(\lambda-\lambda_2)y(\lambda, x)] = a_0\left[x^\lambda + (\lambda+1)x^\lambda \ln x + \frac{1}{(\lambda+3)^2}x^{\lambda+2} - \frac{1}{(\lambda+3)}x^{\lambda+2}\ln x - \frac{1}{(\lambda+5)^2(\lambda+3)^2}x^{\lambda+4} - \frac{2}{(\lambda+5)(\lambda+3)^3}x^{\lambda+4} + \frac{1}{(\lambda+5)(\lambda+3)^2}x^{\lambda+4}\ln x + \cdots\right]$$

Then

$$\begin{aligned} y_2(x) &= \frac{\partial}{\partial\lambda}[(\lambda-\lambda_2)y(\lambda,x)]\bigg|_{\lambda=\lambda_2=-1} \\ &= a_0\left(x^{-1} + 0 + \frac{1}{4}x - \frac{1}{2}x\ln x - \frac{1}{64}x^3 - \frac{2}{32}x^3 + \frac{1}{16}x^3\ln x + \cdots\right) \\ &= -\frac{1}{2}(\ln x)a_0x\left(1 - \frac{1}{8}x^2 + \cdots\right) + a_0\left(x^{-1} + \frac{1}{4}x - \frac{5}{64}x^3 + \cdots\right) \\ &= -\frac{1}{2}(\ln x)y_1(x) + a_0x^{-1}\left(1 + \frac{1}{4}x^2 - \frac{5}{64}x^4 + \cdots\right) \end{aligned} \qquad (1)$$

This is in the form of (*28.9*) with $d_{-1} = -\frac{1}{2}$, $d_0 = a_0$, $d_1 = 0$, $d_2 = \frac{1}{4}a_0$, $d_3 = 0$, $d_4 = \frac{-5}{64}a_0, \ldots$. The general solution is $y = c_1y_1(x) + c_2y_2(x)$.

28.18. Use the method of Frobenius to find one solution near $x = 0$ of $x^2y'' + (x^2 + 2x)y' - 2y = 0$.

Here

$$P(x) = 1 + \frac{2}{x} \quad \text{and} \quad Q(x) = -\frac{2}{x^2}$$

so $x = 0$ is a regular singular point and the method of Frobenius is applicable. Substituting Eqs. (*28.2*) through (*28.4*) into the left side of the differential equation, as given, and combining coefficients of like powers of x, we obtain

$$x^\lambda[(\lambda^2+\lambda-2)a_0] + x^{\lambda+1}[(\lambda^2+3\lambda)a_1 + \lambda a_0] + \cdots + x^{\lambda+n}\{[(\lambda+n)^2 + (\lambda+n) - 2]a_n + (\lambda+n-1)a_{n-1}\} + \cdots = 0$$

Dividing by x^λ, factoring the coefficient of a_n, and equating to zero the coefficient of each power of x, we obtain

$$(\lambda^2+\lambda-2)a_0 = 0 \qquad (1)$$

and, for $n \geq 1$,

$$[(\lambda+n)+2][(\lambda+n)-1]a_n + (\lambda+n-1)a_{n-1} = 0$$

which is equivalent to

$$a_n = -\frac{1}{\lambda+n+2}a_{n-1} \quad (n \geq 1) \qquad (2)$$

From (*1*), the indicial equation is $\lambda^2 + \lambda - 2 = 0$, which has roots $\lambda_1 = 1$ and $\lambda_2 = -2$. Since $\lambda_1 - \lambda_2 = 3$, a positive integer, the solution is given by Eqs. (*28.5*) and (*28.9*). Substituting $\lambda = 1$ into (*2*), we obtain $a_n = [-1/(n+3)]a_{n-1}$, which in turn yields

$$\begin{aligned} a_1 &= -\frac{1}{4}a_0 = -\frac{3!}{4!}a_0 \\ a_2 &= -\frac{1}{5}a_1 = \left(-\frac{1}{5}\right)\left(-\frac{3!}{4!}\right)a_0 = \frac{3!}{5!}a_0 \\ a_3 &= -\frac{1}{6}a_2 = -\frac{3!}{6!}a_0 \end{aligned}$$

and, in general,

$$a_k = \frac{(-1)^k 3!}{(k+3)!}a_0$$

Hence,

$$y_1(x) = a_0 x\left[1 + 3!\sum_{n=1}^{\infty}\frac{(-1)^n x^n}{(n+3)!}\right] = a_0 x\sum_{n=0}^{\infty}\frac{(-1)^n 3!\, x^n}{(n+3)!}$$

which can be simplified to

$$y_1(x) = \frac{3a_0}{x^2}(2 - 2x + x^2 - 2e^{-x}) \tag{3}$$

28.19. Find the general solution near $x = 0$ to the differential equation given in Problem 28.18.

One solution is given by (*3*) in Problem 28.18 for the indicial root $\lambda_1 = 1$. If we try the method of Frobenius with the indicial root $\lambda_2 = -2$, recurrence formula (*2*) of Problem 28.18 becomes

$$a_n = -\frac{1}{n}a_{n-1} \tag{1}$$

which does define all $a_n (n \geq 1)$. Solving sequentially, we obtain

$$a_1 = -a_0 = -\frac{1}{1!}a_0 \qquad a_2 = -\frac{1}{2}a_2 = \frac{1}{2!}a_0$$

and, in general, $a_k = (-1)^k a_0/k!$. Therefore,

$$y_2(x) = a_0 x^{-2}\left[1 - \frac{1}{1!}x + \frac{1}{2!}x^2 + \cdots + \frac{(-1)^k}{k!}x^k + \cdots\right]$$

$$= a_0 x^{-2}\sum_{n=0}^{\infty}\frac{(-1)^n x^n}{n!} = a_0 x^{-2}e^{-x}$$

This is precisely in the form of (*28.9*), with $d_{-1} = 0$ and $d_n = (-1)^n a_0/n!$. The general solution is

$$y = c_1 y_1(x) + c_2 y_2(x)$$

28.20. Find a general expression for the indicial equation of (*28.1*).

Since $x = 0$ is a regular singular point; $xP(x)$ and $x^2Q(x)$ are analytic near the origin and can be expanded in Taylor series there. Thus,

$$xP(x) = \sum_{n=0}^{\infty} p_n x^n = p_0 + p_1 x + p_2 x^2 + \cdots$$

$$x^2 Q(x) = \sum_{n=0}^{\infty} q_n x^n = q_0 + q_1 x + q_2 x^2 + \cdots$$

Dividing by x and x^2, respectively, we have

$$P(x) = p_0 x^{-1} + p_1 + p_2 x + \cdots \qquad Q(x) = q_0 x^{-2} + q_1 x^{-1} + q_2 + \cdots$$

Substituting these two results with Eqs. (*28.2*) through (*28.4*) into (*28.1*) and combining, we obtain

$$x^{\lambda-2}[\lambda(\lambda-1)a_0 + \lambda a_0 p_0 + a_0 q_0] + \cdots = 0$$

which can hold only if

$$a_0[\lambda^2 + (p_0 - 1)\lambda + q_0] = 0$$

Since $a_0 \neq 0$ (a_0 is an arbitrary constant, hence can be chosen nonzero), the indicial equation is

$$\lambda^2 + (p_0 - 1)\lambda + q_0 = 0 \tag{1}$$

28.21. Find the indicial equation of $x^2y'' + xe^xy' + (x^3 - 1)y = 0$ if the solution is required near $x = 0$.

Here

$$P(x) = \frac{e^x}{x} \quad \text{and} \quad Q(x) = x - \frac{1}{x^2}$$

and we have

$$xP(x) = e^x = 1 + x + \frac{x^2}{2!} + \cdots$$
$$x^2Q(x) = x^3 - 1 = -1 + 0x + 0x^2 + 1x^3 + 0x^4 + \cdots$$

from which $p_0 = 1$ and $q_0 = -1$. Using (*1*) of Problem 28.20, we obtain the indicial equation as $\lambda^2 - 1 = 0$.

28.22. Solve Problem 28.9 by an alternative method.

The given differential equation, $3x^2y'' - xy' + y = 0$, is a special case of *Euler's equation*

$$b_nx^ny^{(n)} + b_{n-1}x^{n-1}y^{(n-1)} + \cdots + b_2x^2y'' + b_1xy' + b_0y = \phi(x) \tag{1}$$

where b_j ($j = 0, 1, \ldots, n$) is a constant. Euler's equation can always be transformed into a linear differential equation with *constant coefficients* by the change of variables

$$z = \ln x \quad \text{or} \quad x = e^z \tag{2}$$

It follows from (*2*) and from the chain rule and the product rule of differentiation that

$$\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = \frac{1}{x}\frac{dy}{dz} = e^{-z}\frac{dy}{dz} \tag{3}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(e^{-z}\frac{dy}{dz}\right) = \left[\frac{d}{dz}\left(e^{-z}\frac{dy}{dz}\right)\right]\frac{dz}{dx}$$
$$= \left[-e^{-z}\left(\frac{dy}{dz}\right) + e^{-z}\left(\frac{d^2y}{dz^2}\right)\right]e^{-z} = e^{-2z}\left(\frac{d^2y}{dz^2}\right) - e^{-2z}\left(\frac{dy}{dz}\right) \tag{4}$$

Substituting Eqs. (*2*), (*3*), and (*4*) into the given differential equation and simplifying, we obtain

$$\frac{d^2y}{dz^2} - \frac{4}{3}\frac{dy}{dz} + \frac{1}{3}y = 0$$

Using the method of Chapter 9 we find that the solution of this last equation is $y = c_1e^z + c_2e^{(1/3)z}$. Then using (*2*) and noting that $e^{(1/3)z} = (e^z)^{1/3}$, we have as before,

$$y = c_1x + c_2x^{1/3}$$

28.23. Solve the differential equation given in Problem 28.12 by an alternative method.

The given differential equation, $x^2y'' - xy' + y = 0$, is a special case of Euler's equation, (*1*) of Problem 28.22. Using the transformations (*2*), (*3*), and (*4*) of Problem 28.22, we reduce the given equation to

$$\frac{d^2y}{dz^2} - 2\frac{dy}{dz} + y = 0$$

The solution to this equation is (see Chapter 9) $y = c_1e^z + c_2ze^z$. Then, using (*2*) of Problem 28.22, we have for the solution of the original differential equation

$$y = c_1x + c_2x \ln x$$

as before.

28.24. Find the general solution near $x = 0$ of the *hypergeometric equation*

$$x(1-x)y'' + [C - (A+B+1)x]y' - ABy = 0$$

where A and B are any real numbers, and C is any real nonintegral number.

Since $x = 0$ is a regular singular point, the method of Frobenius is applicable. Substituting, Eqs. (*28.2*) through (*28.4*) into the differential equation, simplifying and equating the coefficient of each power of x to zero, we obtain

$$\lambda^2 + (C-1)\lambda = 0 \tag{1}$$

as the indicial equation and

$$a_{n+1} = \frac{(\lambda+n)(\lambda+n+A+B)+AB}{(\lambda+n+1)(\lambda+n+C)}a_n \tag{2}$$

as the recurrence formula. The roots of (*1*) are $\lambda_1 = 0$ and $\lambda_2 = 1 - C$; hence, $\lambda_1 - \lambda_2 = C - 1$. Since C is not an integer, the solution of the hypergeometric equation is given by Eqs. (*28.5*) and (*28.6*).

Substituting $\lambda = 0$ into (*2*), we have

$$a_{n+1} = \frac{n(n+A+B)+AB}{(n+1)(n+C)}a_n$$

which is equivalent to

$$a_{n+1} = \frac{(A+n)\,(B+n)}{(n+1)(n+C)}a_n$$

Thus

$$a_1 = \frac{AB}{C}a_0 = \frac{AB}{1!C}a_0$$

$$a_2 = \frac{(A+1)(B+1)}{2(C+1)}a_1 = \frac{A(A+1)B(B+1)}{2!C(C+1)}a_0$$

$$a_3 = \frac{(A+2)(B+2)}{3(C+2)}a_2 = \frac{A(A+1)(A+2)B(B+1)(B+2)}{3!C(C+1)(C+2)}a_0$$

....

and $y_1(x) = a_0F(A, B; C; x)$, where

$$F(A, B; C; x) = 1 + \frac{AB}{1!C}x + \frac{A(A+1)B(B+1)}{2!C(C+1)}x^2 + \frac{A(A+1)(A+2)B(B+1)(B+2)}{3!C(C+1)(C+2)}x^3 + \cdots$$

The series $F(A, B; C; x)$ is known as the hypergeometric series; it can be shown that this series converges for $-1 < x < 1$. It is customary to assign the arbitrary constant a_0 the value 1. Then $y_1(x) = F(A, B; C; x)$ and the hypergeometric series is a solution of the hypergeometric equation.

To find $y_2(x)$, we substitute $\lambda = 1 - C$ into (*2*) and obtain

$$a_{n+1} = \frac{(n+1-C)(n+1+A+B-C)+AB}{(n+2-C)(n+1)}a_n$$

or

$$a_{n+1} = \frac{(A-C+n+1)(B-C+n+1)}{(n+2-C)(n+1)}a_n$$

Solving for a_n in terms of a_0, and again setting $a_0 = 1$, it follows that

$$y_2(x) = x^{1-C}F(A-C+1, B-C+1; 2-C; x)$$

The general solution is $y = c_1y_1(x) + c_2y_2(x)$.

Supplementary Problems

In Problems 28.25 through 28.33, find two linearly independent solutions to the given differential equations.

28.25. $2x^2y'' - xy' + (1 - x)y = 0$

28.26. $2x^2y'' + (x^2 - x)y' + y = 0$

28.27. $3x^2y'' - 2xy' - (2 + x^2)y = 0$

28.28. $xy'' + y' - y = 0$

28.29. $x^2y'' + xy' + x^3y = 0$

28.30. $x^2y'' + (x - x^2)y' - y = 0$

28.31. $xy'' - (x + 1)y' - y = 0$

28.32. $4x^2y'' + (4x + 2x^2)y' + (3x - 1)y = 0$

28.33. $x^2y'' + (x^2 - 3x)y' - (x - 4)y = 0$

In Problem 28.34 through 28.38, find the general solution to the given equations using the method described in Problem 28.22.

28.34. $4x^2y'' + 4xy' - y = 0$

28.35. $x^2y'' - 3xy' + 4y = 0$

28.36. $2x^2y'' + 11xy' + 4y = 0$

28.37. $x^2y'' - 2y = 0$

28.38. $x^2y'' - 6xy' = 0$

CHAPTER 29

Some Classical Differential Equations

CLASSICAL DIFFERENTIAL EQUATIONS

Because some special differential equations have been studied for many years, both for the aesthetic beauty of their solutions and because they lend themselves to many physical applications, they may be considered *classical*. We have already seen an example of such an equation, the equation of *Legendre*, in Problem 27.13.

We will touch upon four classical equations: the *Chebyshev* differential equation, named in honor of Pafnuty Chebyshey (1821–1894); the *Hermite* differential equation, so named because of Charles Hermite (1822–1901); the *Laguerre* differential equation, labeled after Edmond Laguerre (1834–1886); and the *Legendre* differential equation, so titled because of Adrien Legendre (1752–1833). These equations are given in Table 29-1 below:

Table 29-1
(Note: $n = 0, 1, 2, 3, \ldots$)

Chebyshev Differential Equation	$(1 - x^2)\, y'' - xy' + n^2 y = 0$
Hermite Differential Equation	$y'' - 2xy' + 2ny = 0$
Laguerre Differential Equation	$xy'' + (1 - x)y' + ny = 0$
Legendre Differential Equation	$(1 - x^2)y'' - 2xy' + n(n + 1)y = 0$

POLYNOMIAL SOLUTIONS AND ASSOCIATED CONCEPTS

One of the most important properties these four equations possess, is the fact that they have *polynomial* solutions, naturally called Chebyshev polynomials, Hermite polynomials, etc.

There are many ways to obtain these polynomial solutions. One way is to employ series techniques, as discussed in Chapters 27 and 28. An alternate way is by the use of *Rodrigues* formulas, so named in honor of O. Rodrigues (1794–1851), a French banker. This method makes use of repeated differentiations (see, for example, Problem 29.1).

These polynomial solutions can also be obtained by the use of *generating functions*. In this approach, infinite series expansions of the specific function "generates" the desired polynomials (see Problem 29.3). It should be noted, from a computational perspective, that this approach becomes more time-consuming the further along we go in the series.

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