Automatic Proof

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Chapter 1

Sister Celine's Method

1.1 Introduction

The subject of computerized proofs of identities begins with the Ph.D. thesis of Sister Mary Celine Fasenmyer (who is often called Sister Celine) at the University of Michigan in 1945. There she developed a method for finding recurrence relations for hypergeometric polynomials directly from the series expansions of the polynomials. The method is quite effective and easily computerized, though it is usually slow in comparison to the method of Zeilberger. Her algorithm is also important because it has yielded general existence theorems for the recurrence relations satisfied by hypergeometric sums. In spite of that, this method have not been used widely because of the lack of tools, such as computer algebra systems, for the necessary calculations.

1.2 Sister Celine's Algorithm

We begin by illustrating her method on a simple sum.

Example 1.1. *Let*

$$f(n) = \sum_k k\binom{n}{k}, \quad n = 1, 2, \ldots,$$

and let's look for the recurrence that $f(n)$ satisfies. To do this we first look for the *recurrence that the summand*

$$F(n,k) = k\binom{n}{k}$$

satisfies. It is a function of two variables (n,k), *so we try to find a recurrence of the form*

$$a(n)F(n,k) + b(n)F(n+1,k) + c(n)F(n,k+1) + d(n)F(n+1,k+1) = 0, \quad (1.1)$$

in which the coefficients a, b, c, d *depend on* n *only, and not on* k.

To find the coefficients, if they exist, we divide (1.1) *through by F(n,k) getting*

$$a + b\frac{F(n+1,k)}{F(n,k)} + c\frac{F(n,k+1)}{F(n,k)} + d\frac{F(n+1,k+1)}{F(n,k)} = 0. \quad (1.2)$$

Substitute $F(n,k) = k\binom{n}{k}$ *in* (1.2) *we get*

$$a + b\frac{n+1}{n+1-k} + c\frac{n-k}{k} + d\frac{n+1}{k} = 0.$$

Put the whole thing over a common denominator. The numerator is, after collecting the powers of k,

$$(d + (c+2d)n + (c+d)n^2) + (a+b-c-d+(a+b-2c-d)n)k + (c-a)k^2 = 0.$$

The coefficient of each power of k *must vanish. This gives us a system of three equations in four unknowns, namely*

$$\begin{aligned} nc + (n+1)d &= 0, \\ (n+1)a + (n+1)b + (-2n-1)c - (n+1)d &= 0, \\ -a + c &= 0, \end{aligned}$$

to solve for a, b, c, d.

Finding a nontrivial solution is now guaranteed simply because the number of unknowns are more than the number of equations. If we actually solve these equations we find that

$$a = -\frac{n+1}{n}\,d, \quad b = 0, \quad c = -\frac{n+1}{n}\,d.$$

We now substitute these values into the recurrence relation (1.1), and we have the desired "k-free" recurrence for the summand $F(n,k)$, namely

$$-\frac{n+1}{n}F(n,k)-\frac{n+1}{n}F(n,k+1)+F(n+1,k+1)=0.$$

$$F(n+1,k+1)=\frac{n+1}{n}F(n,k)+\frac{n+1}{n}F(n,k+1). \tag{1.3}$$

Now sum (1.3) over all integers k, noticing that the coefficients in the recurrence are free of k's, so the summation over k can operate directly on the F in each term. We get the recurrence

$$\begin{aligned} f(n+1) &= \frac{n+1}{n}f(n)+\frac{n+1}{n}f(n) \\ f(n+1) &= 2\frac{n+1}{n}f(n), \quad n=1,2,\ldots, \end{aligned}$$

such that $f(1)=1$. We can now easily find $f(n)$, the desired sum, since

$$\begin{aligned} f(n+1) &= 2\frac{n+1}{n}f(n) \\ &= 2^2\frac{n+1}{n}\frac{n}{n-1}f(n-1) \\ &= 2^n(n+1)f(1) \\ &= 2^n(n+1). \end{aligned}$$

Hence

$$\begin{aligned} f(n) &= 2^{n-1}n \\ \sum_k k\binom{n}{k} &= 2^{n-1}n, \quad n=1,2,\ldots. \end{aligned}$$

□

Definition 1.1. *A nonzero term $F(n,k)$ is called doubly hypergeometric if both*

$$F(n+1,k)/F(n,k) \quad \text{and} \quad F(n,k+1)/F(n,k)$$

are rational functions of n and k.

Now let's discuss her algorithm in general. We are given a sum $f(n) = \sum_k F(n,k)$, where F is doubly hypergeometric. We want to find a recurrence formula for the sum $f(n)$, so for a first step, we will find a recurrence for the summand $F(n,k)$, of the form

$$\sum_{i=0}^{I}\sum_{j=0}^{J} a_{ij}(n)F(n-j,k-i) = 0. \tag{1.4}$$

The complete sequence of steps is the following.

1. Fix trial values of I and J, say $I = J = 1$.

2. Assume the recurrence formula in the form of (1.4), with the coefficients $a_{ij}(n)$ to be determined, if possible.

3. Divide each term of (1.4) by $F(n,k)$, and reduce each ratio $F(n-j,k-i)/F(n,k)$ by simplifying the ratios of the factorials that it contains, so that only rational functions of n and k remain.

4. Place the entire expression over a single common denominator. Then collect the numerator as a polynomial in k.

5. Solve the system of linear equations that results from equating to zero the coefficients of each power of k in the numerator polynomial, for the unknown coefficients $a_{ij}(n)$. If the system has no solution, try the whole thing again with larger values of I and/or J. That is, look for a bigger recurrence.

We will prove below that under suitable hypotheses Sister Celine's algorithm is guaranteed to succeed if I, J are large enough, and the "large enough" can be estimated in advance.

1.3 The Fundamental Theorem

The "Fundamental Theorem" states that every proper hypergeometric term $F(n,k)$ satisfies a recurrence relation of the kind we have found in the previous sections, and it validates the procedure that we have used to find these recurrences in the sense that it guarantees that Sister Celine's method will work if the span of the assumed recurrence is large enough. The theorem also finds explicit precomputable upper bounds on the span.

Definition 1.2. *A function $F(n,k)$ is said to be a proper hypergeometric term if it can be written in the form*

$$F(n,k) = P(n,k)\frac{\prod_{i=1}^{uu}(a_i n + b_i k + c_i)!}{\prod_{i=1}^{vv}(u_i n + v_i k + w_i)!}\, x^k, \tag{1.5}$$

in which x is an indeterminate over, say, the complex numbers, and

1. P is a polynomial,
2. the a's, b's, u's, v's are specific integers, that is to say, they do not contain any additional parameters.
3. the quantities uu and vv are finite, nonnegative, specific integers.

An F of the form (1.5) is well defined at a point (n,k) if none of the numbers $\{a_i n + b_i k + c_i\}_{i=1}^{uu}$ is a negative integer. We will say that $F(n,k) = 0$ if F is well defined at (n,k) and at least one of the numbers $\{u_i n + v_i k + w_i\}_{i=1}^{vv}$ is a negative integer, or $P(n,k) = 0$.

Some examples of proper hypergeometric terms are as follows:

Example 1.2. *The term $\binom{n}{k}2^k$ is proper hypergeometric.*

Solution. The term $\binom{n}{k}2^k$ can be written

$$F(n,k) = \binom{n}{k}2^k = \frac{n!}{k!(n-k)!}2^k,$$