If we identify the above specific F with the general form in (1.5) by taking $P(n,k)=1$, $uu=1, vv=2, x=2, (a_1,b_1,c_1)=(1,0,0)$ and for the two (u,v,w) vectors,

$$(0,1,0),(1,-1,0)$$

So the above F is exactly of the required form.

Example 1.3. *The term $1/(n+3k+1)$ is proper hypergeometric.*

Solution. The term $F(n,k)=1/(n+3k+1)$ is not in proper hypergeometric form. It doesn't contain any of the factorials, and it isn't a polynomial. However, the definition says "...if it can be written in the form ..." This $F(n,k)$ can be written in proper hypergeometric form, even though it was not given to us in that form! All we have to do is to write

$$\frac{1}{n+3k+1}=\frac{(n+3k)!}{(n+3k+1)!}.$$

If we identify the above specific F with the general form in (1.5) by taking $P(n,k)=1$, $uu=1, vv=1, x=1, (a_1,b_1,c_1)=(1,3,0)$ and $(u_1,v_1,w_1)=(1,3,1)$, So the above F is exactly of the required form.

Definition 1.3. *The rising factorial* (rf) *and falling factorial* (ff) *polynomials, for nonnegative integer values of x (the empty product is =1) are defined by*

$$\begin{aligned}\mathrm{rf}(x,y) &= \prod_{j=1}^{x}(y+j)=(y+1)(y+2)\cdots(y+x),\\ \mathrm{ff}(x,y) &= \prod_{j=0}^{x-1}(y-j)=y(y-1)\cdots(y-x+1).\end{aligned}$$

Let $\deg(p)$ denotes the polynomial degree. Now we can state the main theorem.

Theorem 1.1. *Let $F(n,k)$ be a proper hypergeometric term. Then F satisfies a k-free recurrence relation. That is to say, there exist positive integers I,J, and polynomials $a_{ij}(n)$ for $i=0,1,\ldots,I$; $j=0,1,\ldots,J$, not all zero, such that the recurrence*

$$\sum_{i=0}^{I}\sum_{j=0}^{J}a_{ij}(n)F(n-j,k-i)=0 \tag{1.6}$$

holds at every point (n,k) at which $F(n,k) \neq 0$ and all of the values of F that occur in (1.6) are well defined. Furthermore, there is such a recurrence with $(I,J) = (I^*, J^*)$ where

$$J^* = \sum_s |b_s| + \sum_s |v_s|; \quad I^* = 1 + \deg(P) + J^* \left(\left\{ \sum_s |a_s| + \sum_s |u_s| \right\} - 1 \right). \tag{1.7}$$

Note that the recurrence (1.6) is k-$free$ since the coefficients $a_{ij}(n)$ depend only on n, not on k.

Proof. (H.W.) If $F(n,k) = (an + bk + c)!$ then for $i, j \geq 0$ we have

$$\frac{F(n-j,k-i)}{F(n,k)} = \begin{cases} \{(an+bk+c)\cdots(an+bk+c-aj-bi+1)\}^{-1}, & \text{if } aj+bi \geq 0; \\ (an+bk+c+|aj+bi|)\cdots(an+bk+c+1), & \text{if } aj+bi < 0. \end{cases} \tag{1.8}$$

The result is either a polynomial in n and k, or is the reciprocal of such a polynomial, depending on the sign of $aj + bi$. In terms of rf and ff, we can rewrite (1.8) as

$$\frac{F(n-j,k-i)}{F(n,k)} = \begin{cases} 1/\mathrm{ff}(aj+bi, an+bk+c), & \text{if } aj+bi \geq 0; \\ \mathrm{rf}(|aj+bi|, an+bk+c), & \text{if } aj+bi < 0. \end{cases} \tag{1.9}$$

Now consider a function $F(n,k)$ as in (1.5)

$$F(n,k) = P(n,k) \frac{\prod_{s=1}^{uu} (a_s n + b_s k + c_s)!}{\prod_{s=1}^{vv} (u_s n + v_s k + w_s)!} x^k.$$

Let

$$\rho = F(n-j,k-i)/F(n,k)$$

By using (1.9)

$$\rho = \frac{\nu(n,k)}{\delta(n,k)},$$

where

$$\nu(n,k) = P(n-j,k-i) \prod_{\substack{s=1 \\ a_s j + b_s i < 0}}^{uu} \mathrm{rf}(|a_s j + b_s i|, a_s n + b_s k + c_s)$$

$$\prod_{\substack{s=1 \\ u_sj+v_si\geq 0}}^{vv} \text{ff}(u_sj+v_si, u_sn+v_sk+w_s),$$

and

$$\begin{aligned} \delta(n,k) &= P(n,k)x^i \prod_{\substack{s=1 \\ a_sj+b_si\geq 0}}^{uu} \text{ff}(a_sj+b_si, a_sn+b_sk+c_s) \\ & \prod_{\substack{s=1 \\ u_sj+v_si<0}}^{vv} \text{rf}(|u_sj+v_si|, u_sn+v_sk+w_s). \end{aligned} \tag{1.10}$$

Let's assume the recurrence in the form (1.6) and try to solve for the coefficients $a_{ij}(n)$. After dividing by $F(n,k)$, the left side of the assumed recurrence will be

$$\sum_{i=0}^{I}\sum_{j=0}^{J} a_{ij}(n)\frac{\nu_{ij}}{\delta_{ij}}. \tag{1.11}$$

The next step is to collect all of the terms in the sum (1.11) over a single least common denominator. The first thing to notice is that, by (1.10), each and every denominator in (1.11) contains the same factor $P(n,k)$, so $P(n,k)$ will be in the least common denominator that we are constructing.

We introduce the symbol $x^+ = \max(x,0)$, where x is a real number. Then for all real numbers a,b we have

$$\max\{|aj+bi| : aj+bi<0; 0\leq i\leq I; 0\leq j\leq J\} = (-a)^+J + (-b)^+I,$$

and

$$\max\{aj+bi : aj+bi\geq 0; 0\leq i\leq I; 0\leq j\leq J\} = a^+J + b^+I,$$

so the 'x^+' notation is a device that saves the enumeration of many different cases.

Now we can address the question of finding the least common multiple of all of the δ_{ij}'s in (1.11). For each s, a common multiple of all of the falling factorials that appear there will be the one whose first argument is largest, i.e.,

$$\text{ff}((a_s)^+J + (b_s)^+I, a_sn+b_sk+c_s),$$

and a common multiple of all of the rising factorials that appear there will similarly be

$$\mathrm{rf}((-u_s)^+J + (-v_s)^+I, u_s n + v_s k + w_s).$$

Consequently the least common denominator of the expression (1.11), when that expression is thought of as a rational function of k, with n as a parameter, surely divides $P(n,k)$ times

$$\prod_{s=1}^{uu} \mathrm{ff}((a_s)^+J + (b_s)^+I; a_s n + b_s k + c_s) \prod_{s=1}^{vv} \mathrm{rf}((-u_s)^+J + (-v_s)^+I; u_s n + v_s k + w_s). \quad (1.12)$$

Therefore we can clear (1.11) of fractions if we multiply it through by (1.12). The result of multiplying (1.11) through by (1.12) will be the polynomial in k

$$\sum_{i=0}^{I}\sum_{j=0}^{J} a_{ij}(n)\nu_{ij}(n,k)\frac{\Delta}{\delta_{ij}(n,k)}, \quad (1.13)$$

in which Δ is the common denominator in (1.12).

In order to prove the theorem we must show that if I and J are large enough, then the system of linear equations in the unknown a_{ij}'s that one obtains by equating to zero the coefficient of every power of k that appears in (1.13) actually has a nontrivial solution. This will surely happen if the number of unknowns exceeds the number of equations.

Indeed, the number of unknown a_{ij}'s is obviously $(I+1)(J+1)$. The number of equations that they must satisfy is the number of different powers of k that appear in (1.13). We claim that the number of different powers of k that appear there grows only linearly with I and J, that is, as $c_1 I + c_2 J + c_3$, where the c's are independent of I, J. This claim would be sufficient to prove the theorem because then the number of unknowns would grow like IJ, for large I and J, whereas the number of equations would grow only as $c_1 I + c_2 J + c_3$. Hence for large enough I, J the latter would be less than the former.

Since the degree in k of each rising factorial and each falling factorial that appears in (1.13) grows linearly with I, J, and there are only a fixed number of each of them, the degrees in k of all of the ν's, δ's and Δ grow linearly with I, J. Hence the claim is clearly true, and the proof of the main theorem is complete. A more detailed argument, which we omit here, shows that the values I^* and J^* that are in the statement of the theorem are already sufficiently large. □

Example 1.4. *The classical Laguerre polynomials is defined by*

$$L_n(x) = \sum_k (-1)^k \binom{n}{k} \frac{x^k}{k!}, \quad n = 0, 1, 2, \ldots,$$

We'll estimate the order of a two-variable recurrence that is satisfied by the summand

$$F(n,k) = (-1)^k \binom{n}{k} \frac{x^k}{k!}, \quad n = 0, 1, 2, \ldots,$$

which can be written as

$$F(n,k) = \frac{n!}{k!(n-k)!} \frac{1}{k!} (-x)^k, \quad n = 0, 1, 2, \ldots. \tag{1.14}$$

The "Fundamental Theorem", Theorem 1.1 *above, guarantees that this F satisfies a recurrence. To estimate the order of the recurrence, identify the specific F in* (1.14) *with the general form in* (1.5) *above by taking $P(n,k) = 1$, $uu = 1, vv = 3, x = -x, (a_1, b_1, c_1) = (1, 0, 0)$ and for the three (u, v, w) vectors,*

$$(0, 1, 0), (1, -1, 0), (0, 1, 0)$$

Then for the quantitative estimates provided by the theorem, namely the (I^*, J^*) of (1.14)*, we find $J^* = 3, I^* = 4$. Hence there is surely a recurrence for $F(n,k)$ of the form*

$$\sum_{i=0}^{4} \sum_{j=0}^{3} a_{ij}(n) F(n-j, k-i) = 0,$$

in which the a_{ij}'s are polynomials in n, and are not all zero. □