Chapter one

The real numbers

Section 1.1 the algebraic properties of real numbers

Definition: A function $g : A \times A \longrightarrow A$ is said to be a binary operation of .

1.1.1 Algebraic properties of real numbers

On the set of real numbers ($\mathbb{R}$)there are two binary operations, denoted by $+$ and $\cdot$ and called addition and multiplication, respectively. These operations satisfy the following properties:

(A1) $a + b = b + a$ for all $a, b \in \mathbb{R}$ **(commutative property of addition)**

(A2) $(a + b) + c = a + (b + c)$ for all $a, b, c \in \mathbb{R}$ **(ossociative property of addition)**

(A3) there exists an element 0 in $\mathbb{R}$ such that $a + 0 = 0 + a$ for all $a \in \mathbb{R}$ **(existence of 0 element).**

(A4) for each $a \in \mathbb{R}$ there exists an element $-a \in \mathbb{R}$ such that $a + (-a) = (-a) + a = 0$ **(existence of negative element).**

From this conditions we get that (R,+) is commutative group.

(M1) $a \cdot b = b \cdot a$ for all $a, b \in \mathbb{R}$ **(commutative property of multiplication)**

(M2) $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ for all $a, b, c \in \mathbb{R}$ **(ossociative property of multiplication)**

(M3) there exists an element 1distinct from 0 in $\mathbb{R}$ such that $a \cdot 1 = 1 \cdot a$ for all $a \in \mathbb{R}$ **(existence of unit element)**.

(M4) for each $a \neq 0 \in \mathbb{R}$ there exists an element $1/a \in \mathbb{R}$ such that $a \cdot (1/a) = (1/a) \cdot a = 1$ **(existence of reciprocals).**

From this conditions we get that (R-{0},.) is commutative group.

(D) $a \cdot (b + c) = a \cdot b + a \cdot c$ and $(b + c) \cdot a = b \cdot a + c \cdot a$ for all $a, b, c \in \mathbb{R}$ **(distribution property of multiplication over addition).**

Hence (R,+,.) is a field.

Theorem 1.1.2

(a) if z and a are elements of $\mathbb{R}$ such that $z + a = a$, then $z = 0$.

(b) if z and $a \neq 0$ are elements of $\mathbb{R}$ such that $z \cdot a = a$, then $z = 1$.

Proof. (a)

$$0 = a + (-a) = z + a + (-a) = z + 0 = z.$$

(b)

$$1 = \frac{a}{a} = \frac{z \cdot a}{a} = z.$$

■

Theorem 1.1.3

(a) if a and b are elements of $\mathbb{R}$ such that $a + b = 0$, then $b = -a$.

(b) if $a \neq 0$ and b are elements of $\mathbb{R}$ such that $a \cdot b = 1$, then $b = 1/a$.

Proof. (a)

Since $$a + b = 0,$$

then $$(-a) + a + b = 0 + (-a),$$

and hence $$b = -a.$$

(b)

Since $$a \cdot b = 1,$$

then $$(1/a) \cdot a \cdot b = (1/a) \cdot 1,$$

and hence $$b = (1/a).$$

$\square$

Theorem 1.1.4 let $a, b \in \mathbb{R}$ be arbitrary elements of $\mathbb{R}$. Then:

(a) the equation $a + x = b$ has the unique solution $x = (-a) + b$.

(b) the equation $a \cdot x = b$ has the unique solution $x = (1/a) \cdot b$.

Proof. (H.W.)

(a)

Since $$a + x = b$$

If $x = (-a) + b$ satisfy the above equation, then we get what we want

Note that $(-a) + a + b = \big((-a) + a\big) + b = 0 + b = b$

To show that it is the only solution, suppose that x_1 is any solution of this equation, then

$$a + x_1 = b.$$

If we add $-a$ to both sides, we get

$$(-a) + (a + x_1) = (-a) + b$$

By using **(A3)**, **(A4)** and **(A2)** we get

$x_1 = (-a) + b$

Hence

$x_1 = x$

(b)

Since $\qquad a \cdot x = b$

If $\quad x = (1/a) \cdot b$ satisfy the above equation, then we get what we want

Note that $\quad a \cdot (1/a) \cdot b = \Big(\big((1/a)\big) \cdot a\Big) \cdot b = 1 \cdot b = b$

To show that it is the only solution, suppose that x_1 is any solution of this equation, then

$$a \cdot x_1 = b.$$

If we multiplying both sides by $1/a$, we get

$$(1/a) \cdot (a \cdot x_1) = (1/a) \cdot b$$

By using **(M3)**, **(M4)** and **(M2)** we get

$x_1 = (1/a) \cdot b$

Hence

$x_1 = x$

$\blacksquare$

In this three theorems established so far we have considered the properties of addition and multiplication separately. To examine the interplay between the two operations we must employ the distributive property (D). this is illustrated in the next theorem.

Theorem 1.1.5 If a is any element of $\mathbb{R}$ then:

(a) $0 \cdot a = 0$

(b) $(-1)a = -a$

(c) $-(-a) = a$

(d) $(-1) \cdot (-1) = 1$

Proof. (a)

From **(M3)** we know that $a \cdot 1 = a$ then **(D)** and **(A3)** give

$$a + 0 \cdot a = 1 \cdot a + 0 \cdot a = (1 + 0) \cdot a = 1 \cdot a = a$$

Since 0 element is unique, by **theorem 1.1.2(a)** we conclude that $0 \cdot a = 0$

(b) we can use **(D)**with **(M3), (A4),** and part **(a)** above to obtain

$$a + (-1) \cdot a = 1 \cdot a + (-1) \cdot a = \big(1 + (-1)\big) \cdot a = 0 \cdot a = 0$$

Thus from **theorem 1.1.3(a)** we conclude that $(-1)a = -a$

(c) since $(-1) \cdot a + a = 0$ according to Thus from **theorem 1.1.3(a)** we get

$$-(-a) = a$$

(d) in part **(b)** above put $a = -1$. Then

$(-1) \cdot (-1) = -(-1)$

And from **(c)** the assertion follows with $a = 1$.

■

Theorem 1.1.6 Let $a, b, c \in \mathbb{R}$ then:

(a) If $a \neq 0$ then $1/a \neq 0$ and $1/(1/a) = a$.

(b) If $a \cdot b = a \cdot c \cdot$ and $a \neq 0$then $b = c$

(c) If $a \cdot b = 0$ then either $a = 0$ or $b = 0$.

Proof.

(a) Let $a \neq 0$ and suppose $1/a = 0$ then

$a \cdot (1/a) = a \cdot 0 = 0$

This contrary assertion to **(M3)**. Thus $1/a \neq 0$

Now since $(1/a) \cdot a = 1$ by using **theorem 1.1.3(b)** we get that

$$1/(1/a) = a$$

(b) If we multiply both sides of equation $a \cdot b = a \cdot c \cdot$by $1/a$ and used **(M2),** we get

$((1/a) \cdot a) \cdot b = ((1/a) \cdot a) \cdot c \cdot$

Thus $1 \cdot b = 1 \cdot c$ is the same as $b = c \cdot$

(c) assume that $a \neq 0$ and since $a \cdot b = 0 = a \cdot 0$ we apply part **(b)** above to get that

$$b = 0$$

The rest is as the same **(H.M.)**

■

Rational numbers

Definition. Any element of $\mathbb{R}$ can be written in the form a/b where $a, b \in \mathbb{Z}$ and $b \neq 0$ are called rational number which is denoted by $\mathbb{Q}$.

i.e. $\mathbb{Q} = \{a/b,\ b \neq 0,\ a, b \in \mathbb{Z}\}$

Theorem.1.1.7 There does not exist a rational number r such that $r^2 = 2$.

Proof. Suppose that r is rational, then there exist an integer elements $a, b \in \mathbb{Z}$ such that $= a/b$, we assumed that a, b have no common integer factors other than 1, then $(a/b)^2 = 2 \implies a^2 = 2b^2$, we see that a^2 even this implies that a is also even (if not then $a = 2m + 1$ is odd, then $a^2 = 4m^2 + 4m + 1$ is also odd). There for b must be an odd integer, however $a = 2m$ for some integer m because a is even, and hence $4m^2 = 2b^2 \implies 2m^2 = b^2$ it follows that b^2 is even and hence b is also even integer, and we arrived at a contradiction to the fact that no integer is both even and odd so r is not rational number.

Section 1.2 The order properties of $\mathbb{R}$

1.2.1The order properties of $\mathbb{R}$

There is a non empty subset P of $\mathbb{R}$ called the set of strictly positive real numbers, that satisfies the following properties:

(1) If $a, b \in P$ then $a + b \in P$.

(2) If $a, b \in P$ then $a \cdot b \in P$.

(3) If $a \in \mathbb{R}$ then exactly one of the following holds:

$a \in P,$ $\quad$ $a = 0\,,$ $\quad$ $-a \in P\,.$

Definition. If $a \in P$ we say that a is strictly positive real number and write $a > 0$. If $a \in P$ or is 0 we say that a is positive real number and write $a \geq 0$. If $-a \in P$ we say that a is strictly negative real number and write $a < 0$. If $-a \in P$ or is 0 we say that a is negative real number and write $a \leq 0$.

Definition. Let $a, b \in \mathbb{R}$

(1) If $a - b \in P$ we write $a > b$ or $b < a$.

(2) If $a - b \in P \cup \{0\}$ we write $a \geq b$ or $b \leq a$.

Note that we write $a < b < c$ to mean that both $a < b$ and $b < c$ are satisfied. Similarly, $b \geq a$ and $b \leq c$ we shall write $a \leq b \leq c$. Also, if $a \leq b$ and $b < c$ are satisfied, we shall write $a \leq b < c$.

We shall now establish some of $\mathbb{R}$ the basic properties of the order relation on $\mathbb{R}$. These are the familiar “rules of inequalities”

Theorem 1.2.1 Let $a, b, c \in \mathbb{R}$

(a) If $a > b$ and b $> c$, then $a > c$.

(b) Exactly one of the following holds: $a > b$, $a = b$, $a < b$.

(c) If $a \geq b$ and $b \geq a$, then $a = b$.

Proof. (a)

Since $a > b$ and, $b > c$, then $a - b \in P$ and $b - c \in P$ and from order property of $\mathbb{R}$ **(1)** we get

$$(a - b) + (b - c) = a - c \in P$$

Then $a > c$

(b) By order property of $\mathbb{R}$ **(3)** we get

$$a - b \in P, \qquad a - b = 0, \qquad -(a - b) = b - a \in P$$

And hence

$$a > b, \quad a = b, \quad b < a .$$

(c) If $a \neq b$, then $a - b \neq 0$, so from part **(b)** above we get either

$$a - b \in P, \text{ or } b - a \in P$$

And hence that is either $a > b$, or $b > a$

In either case one of the hypotheses is contradicted. Therefore we must have $a = b$.

∎

Theorem 1.2.2

(a) If $a \in \mathbb{R}$ and $a \neq 0$ then $a^2 > 0$.

(b) $1 > 0$.

(c) If $n \in \mathbb{N}$, then $n > 0$.

Proof. (a)

By the order property of $\mathbb{R}$ **(3)** if $a \neq 0$, then either $a \in P,$ or $-a \in P$.

If $a \in P$, then by the order property of $\mathbb{R}$ **(2)**, we have
$a \cdot a = a^2 \in P \implies a^2 > 0$.

Similarly, if $-a \in P$, then $(-a) \cdot (-a) \in P$, we have from **theorem 1.1.5** that

$$(-a) \cdot (-a) = \big((-1)a\big)\big((-1)a\big) = (-1)(-1)\, a^2 = a^2 \in P \implies a^2 > 0$$

(b) since $1 = (1)^2$, part **(a)** implies that $1 > 0$.

(c) we use the induction for the natural number k. If $k = 1$ is just part **(b)**. if we suppose the assertion is true for the natural number $k \implies k > 0$. then $1, k \in P$, we then have $k + 1 \in P$ by the order property **(1)**. Hence the assertion is true for all natural numbers.

■

Theorem 1.2.3 Let $a, b, c, d \in \mathbb{R}$

(a) If $a > b$, then $a + c > b + c$.
(b) If $a > b$ and $c > d$, then $a + c > b + d$,
(c) If $a > b$, and $c > o$, then $ca > cb$.

If $a > b$ and $c < o$, then $ca < cb$

(d) If $a > 0$, then $1/a > 0$
If $a < 0$, then $1/a < 0$.

Proof.

(a) Since $a > b$, then $a - b \in P$,The conclude follows from the fact that

$(a + c) - (b + c) = a - b > 0$

Then $a + c > b + c$

(b) Since $a > b \Rightarrow a - b \in P$, $c > d \Rightarrow c - d \in P$, then
$(a + c) - (b + d) = (a - b) + (c - d) \in P$

Hence

$a + c > b + d$

(c) Since $a > b \Rightarrow a - b \in P$, $c > 0 \Rightarrow c \in P$, then
$(ca) - (cb) = c(a - b) \in P$

Hence

$$ca > cb$$

On the other hand, if $c < o \Rightarrow -c \in P$ so that
$(cb) - (ca) = -c(a - b) \in P$

Hence

$$ca < cb$$

(d) If $a > 0$, then $a \neq 0$ by the order property, so that$1/a \neq 0$, If $1/a < 0$, then from part **(c)** with $c = 1/a$ implies that $1 = a(1/a) < 0$, contradicting **theorem 1.2.2(b)**therefore, we have $1/a > 0$

Similarly if $a < 0$, then we assume that $1/a > 0$, and we have contradicting where $1 = a(1/a) < 0$, hence $1/a < 0$

■

Theorem 1.2.4 If a and b are in $\mathbb{R}$ and $a > b$ then $a > \frac{1}{2}(a+b) > b$.

Proof.

Since $a > b$ it follows from **theorem 1.2.3(a)** that $2a = a + a > a + b$ and also that $a + b > b + b = 2b$, therefore we have

$2a > (a+b) > 2b$

By **theorem 1.2.2(c)**, we have $2 > 0$, therefore it follows $\frac{1}{2} > 0$, then we can get

$$a = \frac{1}{2}(2a) > \frac{1}{2}(a+b) > \frac{1}{2}(2b) = b$$

Hence

$a > \frac{1}{2}(a+b) > b$

■

Corollary 1.2.1 $a \in \mathbb{R}$ and $a > 0$, then $a > \frac{1}{2}a > 0$.

Proof. Take $b = 0$ in **theorem 1.2.4**

■

Theorem 1.2.5 If $a \in \mathbb{R}$ is such that $0 \leq a < b$ for every strictly positive $b \in \mathbb{R}$, then $a = 0$.

Proof.

Suppose that for contrary that $a \neq 0 \implies a > 0$. Then it follows from **corollary 1.2.1** that $a > \frac{1}{2}a > 0$. If we take $b_0 = \frac{1}{2}a$. Then we obtain $a > b_0 > 0$, so that it is contradiction with $a < b$, for every $b > 0$. Since the supposition that $a \neq 0$, leads to a contradiction, we conclude $a = 0$

■

Theorem 1.2.6 If $ab > 0$ then either

(1) $a > 0$, and $b > 0$.

(2) $a < 0$, and $b < 0$.

Proof. Note that $ab > 0$ implies that $a \neq 0$ and $b \neq 0$(since if either$a$ or b is 0 then the product would be 0). Then either $a > 0$ or $a < 0$. If $a > 0$, then $1/a > 0$, and therefore

$$b = 1 \cdot b = \big((1/a)a\big)b = 1/a\,(ab) > 0$$

Then $b > 0$

Similarly, if $a < 0$, then $1/a < 0$, so that

$$b = 1 \cdot b = \big((1/a)a\big)b = 1/a\,(ab) < 0$$

Then $b < 0$

■

Corollary 1.2.2 If $ab < 0$ then either

(1) $a > 0$, and $b < 0$.

(2) $a < 0$, and $b > 0$.

Proof. (H.W.)

■

1.3 Absolute value

Definition: If $a \in \mathbb{R}$ the absolute value of a denoted by $|a|$ is defined by

$$|a| = \begin{cases} a, & a \geq 0 \\ -a, & a < 0 \end{cases}$$

Note that $|a| \geq 0$ for all $a \in \mathbb{R}$, moreover, if $|a| = 0$, then $a = 0$.

Theorem 1.3.1

(a) $|a| = |-a| \quad \forall a \in \mathbb{R}$.

(b) $|ab| = |a||b| \quad \forall a, b \in \mathbb{R}$.

(c) If $c > 0$, then $|a| \leq c \quad \Leftrightarrow \quad -c \leq a \leq c$.

(d) $-|a| \leq a \leq |a| \quad \forall a \in \mathbb{R}$.

(e) $|a + b| \leq |a| + |b| \quad \forall a, b \in \mathbb{R}$.

(f) $\big||a| - |b|\big| \leq |a - b| \quad \forall a, b \in \mathbb{R}$.

(g) $|a - b| \leq |a| + |b| \quad \forall a, b \in \mathbb{R}$.

Proof:

(a) If $a = 0$, then $|0| = 0 = |-0|$. If $a > 0$, then $-a \leq 0$ so that $|a| = -(-a) = |-a|$. If $a < 0$, then $-a > 0$ so that $|-a| = -a = |a|$.

(b) If either a or b is 0, then both $|ab| = 0$ and $|a||b| = 0$. If $a > 0$ and $b > 0$, then $ab > 0$ so that $|ab| = ab = |a||b|$. If $a > 0$ and $b < 0$, then $ab < 0$ so that $|ab| = -ab = a(-b) = |a||b|$. The case $a < 0$ and $b > 0$ is similarly. Finally if $a < 0$ and $b < 0$, then $|ab| = ab = (-a)(-b) = |a||b|$

(c) suppose $|a| \leq c$, then we have both $a \leq c$ and $-a \leq c$, since the latter inequality is equivalent to $-c \leq a$, we have $-c \leq a \leq c$. conversely if $-c \leq a \leq c$. then we have both $a \leq c$ and $-a \leq c$, so that $|a| \leq c$.

(d) take $c = |a|$ in part **(c)** above.

(e) since we know that $-|a| \leq a \leq |a|$ and $-|b| \leq b \leq |b|$, then $-(|a| + |b|) \leq a + b \leq |a| + |b|$. Hence we have $|a + b| \leq |a| + |b|$.

(f) from $|a| \leq |a - b + b| \leq |a - b| + |b|$, we obtain $|a| - |b| \leq |a - b|$. Similarly, from $|b| \leq |b - a + a| \leq |b - a| + |a|$, we obtain $-(|a| - |b|) \leq |a - b|$. Combining these two inequalities we get $\big||a| - |b|\big| \leq |a - b|$.

(g) if we put $-b$ in staid of b in (e) we get what we want.

■

Note that the inequality in **(e)** is called triangle inequality. The distance from a to the origin is $|a|$, and the distance from a to b is $|b - a|$.

Neighborhood

Definition: Let $a \in \mathbb{R}$, then

(i) for $\varepsilon > 0$, the ε-neighborhood of a is the set $N_\varepsilon(a) = \{x \in \mathbb{R}: |x - a| < \varepsilon\}$.

(ii) A neighborhood of a is any set that contains an ε-neighborhood of a for some $\varepsilon > 0$.

Example: give the 2- neighborhood of -1.

Solution: $N_2(-1) = (-1 - 2, -1 + 2) = (-3,1)$

Theorem 1.3.2 Let $a \in \mathbb{R}$. If $x \in \mathbb{R}$ is such that x belongs to every neighborhood of a, then $x = a$.

Proof: since x belongs to $N_\varepsilon(a)$ for every $\varepsilon > 0$, then $x \in (a - \varepsilon, a + \varepsilon)$ for every $\varepsilon > 0$ so that $|x - a| < \varepsilon$ for every $\varepsilon > 0$ and since $0 \le |x - a| < \varepsilon$ for every $\varepsilon > 0$, then from **Theorem 1.2.5** we have $|x - a| = 0$ and hence $x = a$.

■

Examples. (a) Let $U = \{x: 0 < x < 1\}$. If $a \in U$, then $0 < a < 1$; hence, if ε is the smaller of the two numbers a and $1 - a$, we see that $N_\varepsilon(a)$ is an ε-neighborhood of a contained in U. Thus $U$is a neighborhood of each of its points. But if $I = \{x: 0 \le x \le 1\}$, then I is not neighborhood of 0 and 1 because for any $\varepsilon > 0$ there exists a number x satisfying $|x| < \varepsilon$ and $|x - a| < \varepsilon$ (therefore $x \notin I$) i.e. there is no ε-neighborhood of 0 and 1 contained in I.

(b) prove that if $x \in N_\varepsilon(a)$ and $y \in N_\varepsilon(b)$, then $x + y \in N_{2\varepsilon}(a + b)$.

Since $\quad x \in N_\varepsilon(a) \quad \Rightarrow \quad |x - a| < \varepsilon$

$y \in N_\varepsilon(b) \quad \Rightarrow \quad |x - b| < \varepsilon$

By using the T. inequality we get

$$|(x + y) - (a + b)| = |(x - a) + (y - b)| \le |(x - a)| + |(y - b)| < 2\varepsilon$$

Hence $x + y \in N_{2\varepsilon}(a + b)$.

Completeness in $\mathbb{R}$

Definition: Let S be a subset of $\mathbb{R}$.

(1) An element $a \in \mathbb{R}$, is said to be an upper bound of S if $a \ge s, \quad \forall s \in S$.

(2) An element $b \in \mathbb{R}$, is said to be a lower bound of S if $b \le s, \quad \forall s \in S$.

Example:(a) Find u. b. and l. b. to the set $S = (-2{,}5)$.

Since $6 \ge s, \quad \forall s \in S$, then $a = 6$ is an u. b. of the set $S = (-2{,}5)$. So that $-10 \le s, \quad \forall s \in S$, then $b = -10$ is a l. b. of the set $S = (-2{,}5)$.

(b) Find u. b. and l. b. to the set $S = \{-13\} \cup (-1{,}3) \cup \{8\}$.

Since $9 \ge s, \ \forall s \in S$, then $a = 9$ is an u. b. of the set $S = \{-13\} \cup (-1{,}3) \cup \{8\}$. So that $-100 \le s, \quad \forall s \in S$, then $b = -100$ is a l. b. of the set $S = \{-3\} \cup (-1{,}3) \cup \{8\}$.

Remark: we say that a subset S of $\mathbb{R}$ is bounded above if it has an upper bound. Similarly if a set S of $\mathbb{R}$ has a lower bound we say it is bounded below. if a set S of $\mathbb{R}$ has both an upper bound and a lower bound we say it is bounded. if a set S of $\mathbb{R}$ don't has either an upper bound or a lower bound we say it is unbounded.

Definition: Let S be a subset of $\mathbb{R}$.

(1) If S is bounded above, then an upper bound is said to be a supremum (denoted by Sup(S)) (or a least upper bound) of S if it is less than every other upper bound of S.

i.e. an upper bound $u \in \mathbb{R}$ of S, is said to be supremum if for every anther upper bound $v \in \mathbb{R}$, we have $u \le v$.

(2) If S is bounded below, then a lower bound is said to be a infimum (denoted by inf(S)) (or a greatest lower bound) of S if it is greatest than every other lower bound of S.

i.e. a lower bound $u \in \mathbb{R}$ of S, is said to be infimum if for every anther lower bound $v \in \mathbb{R}$, we have $u \ge v$.

Example: Find Sup(S) and inf(S) for the following sets

(1) S=(-1,5) **(2)** S=[0,2) **(3)** S={1,2,4,7,8}

Solution:

(1) Sup(S)=5, inf(S)=-1

(2) Sup(S)=2, inf(S)=0

(3) Sup(S)=8, inf(S)=1

Remark: If inf(S) $\in$ S, then inf(S)=min(S). similarly If Sup(S) $\in$ S, then Sup(S)=max(S).

Lemma 1.3.1: An upper bound u of a non-empty set S of $\mathbb{R}$ is the supremum of S if and only if each $\varepsilon > 0$ there exists $s_\varepsilon \in S$ such that $u - \varepsilon < s_\varepsilon$.

Proof: (H.W.)

Suppose that u is u. b. of S and each $\varepsilon > 0$ there exists $s_\varepsilon \in S$ such that $u - \varepsilon < s_\varepsilon$, wanted $u = \sup(S)$, if u is the only u. b., then $u = \sup(S)$. o. w. Let $v = \sup(S) \implies u \neq v$, then either $u > v$ or $u < v$. If $u > v \implies u - v > 0$, if we take $\varepsilon = u - v$, hence there exists $s_\varepsilon \in S \ni u - \varepsilon < s_\varepsilon \implies u - (u - v) = v < s_\varepsilon$, this contradiction since $v = \sup(S)$. Hence $u < v$, then $u = \sup(S)$.

Conversely suppose that $u = \sup(S)$. And want each $\varepsilon > 0$ there exists $s_\varepsilon \in S$ such that $u - \varepsilon < s_\varepsilon$. Directly from definition of supremum.

■

The supremum property of $\mathbb{R}$

Every non-empty set of real numbers that has an upper bound has a supremum.

Note that the analogous property of infima can be readily established.

Example: Let $S \neq \phi \in \mathbb{R}$ which is bounded above. And let a in R, define $a + S = \{a + s : s \in S$. Prove that

$$\sup(a + S) = a + \sup(S)$$

Proof:

Suppose that $u = \sup(S) \;\; u \geq x \;\; \forall x \in S$, then $a + u \geq a + x \quad \forall x \in S$

Therefore $a + u$ is an upper bound for $a + S$, consequently, we have $\sup(a + S) \leq a + u$, if v is any upper bound of the set $a + S$, then $a + x \leq v \;\; \forall x \in S \quad \Longrightarrow x \leq v - a \quad \forall x \in S$, then $v - a$ is an upper bound for S, and since $u = \sup(S)$, then implies that $u \leq v - a \;\; \Longrightarrow a + u \leq v$, then we have that $\sup(a + S) = a + u$, but $u = \sup(S)$, hence

$\sup(a + S) = a + u = a + \sup(S)$

■

Archimedean property

If $x \in \mathbb{R}$ then there exists $n_x \in \mathbb{N}$ such that $x < n_x$.

Proof: If the conclusion fails (i.e.$\forall\, n_x \in \mathbb{N}$ such that,$x \geq n_x$). Then x is an upper bound of $\mathbb{N}$. Therefore by the supremum property, the non-empty set $\mathbb{N}$ has a supremum $u \in \mathbb{R}$. Since $u - 1 < u$, it is follows from **lemma 1.3.1**that there exists $m \in \mathbb{N}$ such that $u - 1 < m$. But then $u < m + 1$, and since $m + 1 \in \mathbb{N}$, this contradicts the assumption that u is an upper bound.

■

Corollary 1.3.1 let y and z be strictly positive real numbers. Then:

(a) There exists $n \in \mathbb{N}$ such that $z < ny$.

(b) There exists $n \in \mathbb{N}$ such that $0 < 1/n < y$.

(c) There exists $n \in \mathbb{N}$ such that $n - 1 \leq z < y$.

Proof: (H.W.)

(a) since $x = z/y > 0$, then from Archimedean property there exists $n \in \mathbb{N}$, such that $x = z/y < n \;\; \Rightarrow \;\; z < ny$.

(b) setting $z = 1$in (a) above gives us $1 < ny$, and since $n \in \mathbb{N} \;\; \Rightarrow \;\; 1/n > 0$, hence $0 < 1/n < y$.

(c) the Archimedean property assures us that the subset $\{m \in \mathbb{N}: z < m\}$ of $\mathbb{N}$ is not empty. Let be the least element of this set then we have $n - 1 \leq z < y$.

■

Density of rational numbers in $\mathbb{R}$.

The density Theorem. If x and y are real numbers with $x < y$ then there exists a rational number r such that $x < r < y$.

Proof: suppose that $x > 0$, by the **Archimedean property** there exists $n \in \mathbb{N}$ such that $n > 1/y - x$. For such an n. We have $ny - nx > 1$. Applying **corollary 1.3.1(c)** to $nx > 0$, we obtain $m \in \mathbb{N}$ such that $m - 1 \leq nx < m$, this m also satisfies $m < ny$, (because $ny - nx > 1$). Since $m - 1 \leq nx < m \implies m \leq nx + 1$ and then $m \leq nx + 1 < ny$. Thus, we have $nx \leq m < ny \implies x \leq m/n < y$, so that $r = m/n$, is a rational number satisfying $x < r < y$.

■

Corollary 1.3.2 If x and y are real numbers with $x < y$ then there exists an irrational number z such that $x < z < y$.

Proof:

Applying the Density theorem to the real number $x/\sqrt{2}$ and $y/\sqrt{2}$ we get that there exists a rational number r such that $x/\sqrt{2} < z < y/\sqrt{2}$, then $z = r\sqrt{2}$ is irrational number satisfying $x < z < y$.

■

Section 1.4 Intervals and Cluster Point

Intervals in $\mathbb{R}$

If $a, b \in \mathbb{R}$, we defined

1- The open interval (a, b) to be the set $(a, b) = \{x \in \mathbb{R}: a < x < b$.

2- The closed interval $[a, b]$ to be the set $[a, b] = \{x \in \mathbb{R}: a \leq x \leq b$.

3- The half open (half closed) interval to be the set

$[a, b) = \{x \in \mathbb{R}: a \leq x < b$. and $(a, b] = \{x \in \mathbb{R}: a < x \leq b$.

4- The open rays (a, ∞) and $(-\infty, b)$ to be the sets

$(a, \infty) = \{x \in \mathbb{R}: x > a\}$. And $(-\infty, b) = \{x \in \mathbb{R}: x < b\}$

5- The closed rays $[a, \infty)$ and $(-\infty, b]$ to be the sets

$[a, \infty) = \{x \in \mathbb{R}: x \geq a\}$. And $(-\infty, b] = \{x \in \mathbb{R}: x \leq b\}$

6- $\mathbb{R} = (-\infty, \infty)$.

7- $(a, a) = \phi$ and $[a, a] = a$.

Definition: we say that a sequence of intervals $\mathrm{I}_n, n \in \mathbb{N}$ is nested if the following chain of inclusions holds:

$\mathrm{I}_1 \supseteq \mathrm{I}_2 \supseteq \mathrm{I}_3 \supseteq \cdots \supseteq \mathrm{I}_{n-1} \supseteq \mathrm{I}_n \supseteq \mathrm{I}_{n+1} \supseteq \cdots$

Example: if $\mathrm{I}_n = [-1/n, 1/n], n \in \mathbb{N}$, then $\mathrm{I}_1 \supseteq \mathrm{I}_2 \cdots \supseteq \mathrm{I}_{n-1} \supseteq \mathrm{I}_n \cdots$ for each $n \in \mathbb{N}$ so that the intervals are nested, the element 0 belongs to all I_n and the Archimedean property can be used to prove that 0 is the only such element.