III

THE RIEMANN–LIOUVILLE FRACTIONAL INTEGRAL

1. INTRODUCTION

After the lengthy justifications of Chapter II, we begin our mathematical development of the fractional calculus. We start with a formal definition of the Riemann–Liouville fractional integral, carefully delineating the class of functions to which this fractional operator may be applied. Numerous examples, some trivial and some not so elementary, are given and discussed. This analysis provides a convenient vehicle for introducing certain new functions such as $E_t(\nu, a)$, $C_t(\nu, a)$, $S_t(\nu, a)$ that play a forward role in the fractional calculus and fractional differential equations. (Properties of these functions are examined in some detail in Appendix C.)

Certain techniques are developed that enable us to find fractional integrals of more complicated functions. In Section III-4 we consider the Dirichlet formula and analyze some of its consequences. Most prominent is its use in the proof of the law of exponents for fractional integrals. That is, we shall show that ${}_0D_t^{-\mu}({}_0D_t^{-\nu}) = {}_0D_t^{-\mu-\nu}$ for all positive μ and ν (Theorem 1). It also will be used to obtain the fractional integrals of certain nonelementary functions.

In later sections we examine the relations that exist between (ordinary) derivatives of fractional integrals and fractional integrals of derivatives. Many ancillary results in the theory of the fractional calculus may be deduced from these theorems. The penultimate section is devoted to the problem of finding the Laplace transform of fractional integrals, together with the inevitable consequences. The Laplace transform frequently will be exploited in remaining chapters, especially in our study of fractional differential equations. In the final section we discuss Leibniz's formula for fractional integrals and give some interesting applications of this rule.

2. DEFINITION OF THE FRACTIONAL INTEGRAL

As we have stated before, our objective is to investigate various aspects of the Riemann–Liouville fractional integral. We begin with a formal definition (see Definition 1 below).

Let X be a positive number and let f be continuous on $[0, X]$. Then if $\nu \geqq 1$,

$$\int_0^t (t - \xi)^{\nu-1} f(\xi)\, d\xi \tag{2.1}$$

exists as a Riemann integral for all $t \in [0, X]$. Of course, (2.1) will exist under more general conditions. For example, if f is continuous on $(0, X]$ and behaves like t^λ for $-1 < \lambda < 0$ in a neighborhood of the origin and/or if $0 < \operatorname{Re} \nu < 1$, then (2.1) exists as an improper Riemann integral. The following definition, however, is sufficiently broad for our purposes.

Definition 1. Let $\operatorname{Re} \nu > 0$ and let f be piecewise continuous on $J' = (0, \infty)$ and integrable on any finite subinterval of $J = [0, \infty)$. Then for $t > 0$ we call

$${}_0D_t^{-\nu} f(t) = \frac{1}{\Gamma(\nu)} \int_0^t (t - \xi)^{\nu-1} f(\xi)\, d\xi \tag{2.2}$$

the Riemann–Liouville fractional integral of f of order ν.

Let us discuss this definition. As we have observed above, (2.2) is an improper integral if $0 < \operatorname{Re} \nu < 1$. We require f to be piecewise continuous only on $J' = (0, \infty)$ (the interval J excluding the origin) to accommodate functions that behave like $\ln t$ or t^μ (for $-1 < \mu < 0$) in a neighborhood of the origin. We shall denote by $\mathbf{C}$ the class of functions described in Definition 1. [One readily may generalize $\mathbf{C}$ to include, for example, such functions as $f(\xi) = |\xi - a|^\lambda$, $\lambda > -1$, $0 < a < t$. We seldom shall have occasion to do so.]

For example, if $f(t) = t^{\mu}$ with $\mu > -1$, then [see (II-6.3), p. 36]

$$ {}_0D_t^{-\nu} t^{\mu} = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\nu+1)} t^{\mu+\nu}, \qquad t > 0 \tag{2.3} $$

[since (2.2) is now essentially the beta function]. Because $\mu + \operatorname{Re} \nu$ may be negative, we see from this example why we must include the caveat $t > 0$ in our definition of the fractional integral. [Of course, if $\mu \geqq 0$, then (2.3) is continuous on J.] To avoid minor mathematical complications not related to the fractional calculus, and with little loss of generality, we shall, as a practical matter, assume that ν is real. Occasionally, we indicate that certain formulas are valid for $\operatorname{Re} \nu > 0$ rather than just for $\nu > 0$. A discussion of fractional operators when ν is purely imaginary may be found in [19].

If we write (2.2) as the Stieltjes integral

$$ {}_0D_t^{-\nu} f(t) = \frac{1}{\Gamma(\nu+1)} \int_0^t f(\xi)\, d\alpha(\xi), $$

where

$$ \alpha(\xi) = -(t-\xi)^{\nu} \tag{2.4} $$

is a (continuous) monotonic increasing function of ξ on $[0, t]$, then if f is continuous on $[0, t]$, the first mean value theorem for integrals [45, p. 107] implies that

$$ \int_0^t f(\xi)\, d\alpha(\xi) = f(x) t^{\nu} $$

for some $x \in [0, t]$. Hence

$$ \lim_{t \to 0} {}_0D_t^{-\nu} f(t) = 0. \tag{2.5} $$

If f is not continuous (but still of class **C**), then (2.5) need not be true. In fact, we see from (2.3) with $\nu > 0$, $\mu > -1$, that

$$ \lim_{t \to 0} {}_0D_t^{-\nu} t^{\mu} = \begin{cases} 0, & \mu + \nu > 0 \\ \Gamma(\mu+1), & \mu + \nu = 0 \\ \infty, & \mu + \nu < 0. \end{cases} $$

Furthermore, we also conclude from (2.3) that even the continuity of f at the origin does not guarantee the differentiability of ${}_0D_t^{-\nu}f(t)$ at $t = 0$. (For example, let $\mu > 0$ and $\mu + \nu < 1$.)

At times it may be expedient to consider certain subclasses of **C**. For instance, in Chapter IV we introduce a class of functions that includes functions of the form

$$t^{\lambda}\eta(t)$$

where $\lambda > -1$ and $\eta(t)$ is analytic. At other times we shall find it convenient to take the Laplace transform of the fractional integral. In such cases we require that f be of exponential order. Since we mainly shall be considering integrals of the form (2.2), the notation will be simplified by dropping the subscripts 0 and t on ${}_0D_t^{-\nu}$, as was done in Section II-7. Occasionally, we shall use them for emphasis, or if there is a possibility of ambiguity, or if we wish to consider a fractional integral whose lower limit is not zero.

3. SOME EXAMPLES OF FRACTIONAL INTEGRALS

Before we embark on a theoretical analysis of the fractional integral, let us calculate the fractional integrals of a few elementary functions. We already have shown in (2.3) that

$$D^{-\nu}t^{\mu} = \frac{\Gamma(\mu + 1)}{\Gamma(\mu + \nu + 1)}t^{\mu+\nu}, \qquad \nu > 0, \quad \mu > -1, \quad t > 0. \tag{3.1}$$

In particular, if $\mu = 0$, the fractional integral of a constant K of order ν is

$$D^{-\nu}K = \frac{K}{\Gamma(\nu + 1)}t^{\nu}, \qquad \nu > 0. \tag{3.2}$$

Perhaps the reader may have wondered why we did not give a few additional examples of fractional integrals. The answer is simple—fractional integrals, even of such elementary functions as exponentials and sines and cosines, lead to higher transcendental functions—as we shall now demonstrate.

Suppose that

$$f(t) = e^{at}$$

where a is a constant. Certainly, e^{at} is of class **C**, and by Definition 1,

$$D^{-\nu}e^{at} = \frac{1}{\Gamma(\nu)}\int_0^t (t-\xi)^{\nu-1}e^{a\xi}\,d\xi, \qquad \nu > 0. \tag{3.3}$$

If we make the change of variable $x = t - \xi$, (3.3) becomes

$$D^{-\nu}e^{at} = \frac{e^{at}}{\Gamma(\nu)}\int_0^t x^{\nu-1}e^{-ax}\,dx, \qquad \nu > 0. \tag{3.4}$$

Clearly, (3.4) is not an elementary function. But it *is* closely related to the transcendental function known as the incomplete gamma function [(B-2.19), p. 300, Section C-2]. For Re $\nu > 0$ the incomplete gamma function $\gamma^*(\nu, t)$ may be defined as

$$\gamma^*(\nu, t) = \frac{1}{\Gamma(\nu)t^\nu}\int_0^t \xi^{\nu-1}e^{-\xi}\,d\xi. \tag{3.5}$$

Thus we may write (3.4) as

$$D^{-\nu}e^{at} = t^\nu e^{at}\gamma^*(\nu, at). \tag{3.6}$$

Since the right-hand side of (3.6) is the fractional integral of an exponential, it is not surprising that this function frequently arises in the study of the fractional calculus. We shall call it $E_t(\nu, a)$,

$$E_t(\nu, a) = t^\nu e^{at}\gamma^*(\nu, at). \tag{3.7}$$

Some of the elementary properties of γ^* and E_t are examined in Appendix C.

A direct application of the definition of the fractional integral leads to

$$D^{-\nu}\cos at = \frac{1}{\Gamma(\nu)}\int_0^t \xi^{\nu-1}\cos a(t-\xi)\,d\xi, \qquad \nu > 0 \tag{3.8}$$

and

$$D^{-\nu}\sin at = \frac{1}{\Gamma(\nu)}\int_0^t \xi^{\nu-1}\sin a(t-\xi)\,d\xi, \qquad \nu > 0. \tag{3.9}$$

We find it convenient to define the right-hand sides of (3.8) and (3.9) as $C_t(\nu, a)$ and $S_t(\nu, a)$, respectively. Properties of these functions also are studied in Appendix C.

Thus from (3.7), (3.8), and (3.9) we have for $\nu > 0$ the compact formulas

$$\begin{aligned} D^{-\nu} e^{at} &= E_t(\nu, a) \\ D^{-\nu} \cos at &= C_t(\nu, a) \\ D^{-\nu} \sin at &= S_t(\nu, a). \end{aligned} \tag{3.10}$$

In the special case $\nu = \frac{1}{2}$,

$$\begin{aligned} D^{-1/2} e^{at} &= E_t(\tfrac{1}{2}, a) \\ &= a^{-1/2} e^{at} \operatorname{Erf}(at)^{1/2}, \end{aligned} \tag{3.11}$$

where Erf x is the error function (B-2.25), p. 301. Also,

$$\begin{aligned} D^{-1/2} \cos at &= C_t(\tfrac{1}{2}, a) \\ &= \sqrt{\frac{2}{a}}\, [(\cos at) C(x) + (\sin at) S(x)] \end{aligned} \tag{3.12}$$

and

$$\begin{aligned} D^{-1/2} \sin at &= S_t(\tfrac{1}{2}, a) \\ &= \sqrt{\frac{2}{a}}\, [(\sin at) C(x) - (\cos at) S(x)], \end{aligned} \tag{3.13}$$

where

$$x = \sqrt{\frac{2at}{\pi}}$$

and $C(x)$ and $S(x)$ are the Fresnel integrals (B-2.27) and (B-2.28), p. 301.

Simple trigonometric identities may be used to calculate other fractional integrals of trigonometric functions. For example, from $\cos 2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$,

$$D^{-\nu} \cos^2 at = \frac{t^\nu}{2\Gamma(\nu + 1)} + \frac{1}{2} C_t(\nu, 2a) \tag{3.14}$$

and

$$D^{-\nu}\sin^2 at = \frac{t^\nu}{2\Gamma(\nu+1)} - \frac{1}{2}C_t(\nu, 2a). \tag{3.15}$$

We consider some slightly more complicated functions. Suppose that

$$f(t) = (a-t)^\lambda, \qquad a > t > 0.$$

Then $f \in \mathbf{C}$, and by definition,

$$D^{-\nu}(a-t)^\lambda = \frac{1}{\Gamma(\nu)}\int_0^t (t-\xi)^{\nu-1}(a-\xi)^\lambda\, d\xi,$$
$$\operatorname{Re}\nu > 0, \quad a > t > 0. \tag{3.16}$$

If we make the bilinear transformation

$$x = \frac{t-\xi}{a-\xi}$$

in the integrand of (3.16), we obtain

$$D^{-\nu}(a-t)^\lambda = \frac{(a-t)^{\lambda+\nu}}{\Gamma(\nu)}\int_0^{t/a} x^{\nu-1}(1-x)^{-\nu-\lambda-1}\, dx.$$

But the integral above is just the incomplete beta function (B-2.24), p. 300. Thus

$$D^{-\nu}(a-t)^\lambda = \frac{1}{\Gamma(\nu)}(a-t)^{\lambda+\nu}\mathrm{B}_{t/a}(\nu, -\lambda-\nu). \tag{3.17}$$

If, in particular, $a = 1$, $\nu = \frac{1}{2}$, and $\lambda = -\frac{1}{2}$, direction integration leads to

$$D^{-1/2}\frac{1}{\sqrt{1-t}} = \frac{1}{\sqrt{\pi}}\ln\frac{1+\sqrt{t}}{1-\sqrt{t}}, \qquad 0 < t < 1. \tag{3.18}$$

As our next example we consider the logarithm. Certainly, $\ln t$ is of class $\mathbf{C}$, and its fractional integral of order ν is

$$D^{-\nu}\ln t = \frac{1}{\Gamma(\nu)}\int_0^t (t-\xi)^{\nu-1}\ln\xi\, d\xi, \qquad \nu > 0.$$

If we make the change of variable $\xi = tx$, then

$$D^{-\nu} \ln t = \frac{t^{\nu}}{\Gamma(\nu+1)} \ln t + \frac{t^{\nu}}{\Gamma(\nu)} \int_0^1 (1-x)^{\nu-1} \ln x \, dx. \quad (3.19)$$

But from [12, p. 538],

$$\int_0^1 x^{\mu-1}(1-x)^{\nu-1} \ln x \, dx = \mathrm{B}(\mu, \nu)[\psi(\mu) - \psi(\mu+\nu)],$$
$$\operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > 0, \quad (3.20)$$

where B is the beta function and ψ is the digamma function (B-2.11), p. 299. Thus if we let $\mu = 1$ in (3.20),

$$D^{-\nu} \ln t = \frac{t^{\nu}}{\Gamma(\nu+1)}[\ln t - \gamma - \psi(\nu+1)], \quad (3.21)$$

where γ is Euler's constant.

If in particular $\nu = \frac{1}{2}$, then

$$\psi\left(\tfrac{3}{2}\right) = 2 - \gamma - \ln 4$$

[21, p. 15] and

$$D^{-1/2} \ln t = \frac{2t^{1/2}}{\sqrt{\pi}}[\ln 4t - 2]. \quad (3.22)$$

More generally, from (3.20) we have

$$D^{-\nu}[t^{\lambda} \ln t] = \frac{\Gamma(\lambda+1)t^{\lambda+\nu}}{\Gamma(\lambda+\nu+1)}[\ln t + \psi(\lambda+1) - \psi(\lambda+\nu+1)],$$
$$\lambda > -1, \quad \nu > 0 \quad (3.23)$$

and with $\nu = \frac{1}{2}$ and $\lambda = -\frac{1}{2}$,

$$D^{-1/2}[t^{-1/2} \ln t] = \sqrt{\pi} \ln \frac{t}{4}$$

[see (B-2.13), p. 299].

Another function, which we shall encounter in our future work, is $f(t) = e^{-1/t}$. [If we define $f(0)$ as zero, all the derivatives of f vanish at the origin. Thus f is not analytic at $t = 0$.] We shall calculate the fractional integral in the more general case where $f(t) = t^{\lambda}e^{-a/t}$, $\lambda > -1$. By definition

$$D^{-\nu}\left[t^{\lambda}e^{-a/t}\right] = \frac{1}{\Gamma(\nu)}\int_0^t (t-\xi)^{\nu-1}\xi^{\lambda}e^{-a/\xi}\,d\xi$$

for $\nu > 0$ and $t > 0$. The change of variable of integration

$$\xi = \frac{t}{x+1}$$

immediately leads to

$$D^{-\nu}\left[t^{\lambda}e^{-a/t}\right] = t^{\lambda+\nu}e^{-a/t}U(\nu, -\lambda, a/t) \tag{3.24}$$

for $\nu > 0$, $\lambda > -1$, $t > 0$. If $\operatorname{Re} a > 0$, then U has the integral representation of (B-4.12), p. 305.

Our ability to calculate explicitly the fractional integral of a function f frequently depends on our proficiency in performing the integration

$$\int_0^t (t-\xi)^{\nu-1}f(\xi)\,d\xi, \qquad \nu > 0. \tag{3.25}$$

However, because of the nature of the kernel $(t-\xi)^{\nu-1}$ in (3.25), it is possible to develop certain analytical techniques that allow us to calculate the fractional integral of a large class of functions with minimal effort. We discuss one such technique now.

The procedure we have in mind will allow us to express the fractional integral of an integral power of t times a function $f(t)$ in terms of fractional integrals of f. Using this argument we may show, for example, that

$$D^{-\nu}\left[te^{at}\right] = tE_t(\nu, a) - \nu E_t(\nu+1, a). \tag{3.26}$$

If $f \in \mathbf{C}$, then from Definition 1, p. 45,

$$D^{-\nu}[tf(t)] = \frac{1}{\Gamma(\nu)}\int_0^t (t-\xi)^{\nu-1}[\xi f(\xi)]\,d\xi, \qquad \nu > 0. \tag{3.27}$$

If we replace the term in brackets in the integrand of (3.27) by the identity

$$[t - (t - \xi)]f(\xi)$$

(i.e., we have added and subtracted t), then (3.27) becomes

$$D^{-\nu}[tf(t)] = tD^{-\nu}f(t) - \nu D^{-\nu-1}f(t). \tag{3.28}$$

In the case $f(t) = e^{at}$, formula (3.28) becomes (3.26) [see (3.10)].
Similarly, (3.28) implies that

$$D^{-\nu}[t\cos at] = tC_t(\nu, a) - \nu C_t(\nu+1, a), \qquad \nu > 0 \tag{3.29}$$

and

$$D^{-\nu}[t\sin at] = tS_t(\nu, a) - \nu S_t(\nu+1, a), \qquad \nu > 0. \tag{3.30}$$

Equation (3.28) may readily be generalized. For if p is a nonnegative integer, then

$$D^{-\nu}[t^p f(t)] = \frac{1}{\Gamma(\nu)}\int_0^t (t-\xi)^{\nu-1}[\xi^p f(\xi)]\,d\xi, \qquad \nu > 0 \tag{3.31}$$

and

$$\xi^p = [t - (t-\xi)]^p = \sum_{k=0}^{p} (-1)^k \binom{p}{k} t^{p-k}(t-\xi)^k.$$

Substituting this expression in (3.31), we obtain

$$\begin{aligned} D^{-\nu}[t^p f(t)] &= \frac{1}{\Gamma(\nu)}\sum_{k=0}^{p}(-1)^k\binom{p}{k}t^{p-k}\int_0^t (t-\xi)^{\nu+k-1}f(\xi)\,d\xi \\ &= \frac{1}{\Gamma(\nu)}\sum_{k=0}^{p}(-1)^k\binom{p}{k}\Gamma(\nu+k)t^{p-k}D^{-(\nu+k)}f(t). \end{aligned} \tag{3.32}$$

Using (B-2.6), p. 298, we also may write (3.32) as

$$D^{-\nu}[t^p f(t)] = \sum_{k=0}^{p}\binom{-\nu}{k}[D^k t^p][D^{-\nu-k}f(t)]. \tag{3.33}$$

For example,

$$D^{-\nu}[t^p e^{at}] = \frac{1}{\Gamma(\nu)} \sum_{k=0}^{p} (-1)^k \binom{p}{k} \Gamma(\nu + k) t^{p-k} E_t(\nu + k, a). \quad (3.34)$$

As we develop further techniques we shall be able to find fractional integrals of still more complicated functions. For example, we show in the next section that for $\nu > 0$ and $\mu > -1$,

$$D^{-\nu} E_t(\mu, a) = E_t(\mu + \nu, a). \quad (3.35)$$

Now let us give a few examples of fractional integrals when the lower limit of integration is not necessarily zero. Consider, then,

$${}_cD_t^{-\nu} f(t) = \frac{1}{\Gamma(\nu)} \int_c^t (t - \xi)^{\nu - 1} f(\xi)\, d\xi, \qquad \nu > 0, \quad 0 \leqq c < t, \quad (3.36)$$

where f is of class **C** on $[c, \infty)$.

The change of variable

$$\xi = t(1 - x)$$

in (3.36) leads to

$${}_cD_t^{-\nu} f(t) = \frac{t^\nu}{\Gamma(\nu)} \int_0^\tau x^{\nu - 1} f(t - tx)\, dx, \quad (3.37)$$

where

$$\tau = \frac{t - c}{t}. \quad (3.38)$$

For example, suppose that

$$f(t) = t^\mu,$$

where $\mu > -1$ if $c = 0$, and μ is arbitrary if $c > 0$. Substitution in (3.37) leads to

$${}_cD_t^{-\nu} t^\mu = \frac{t^{\mu + \nu}}{\Gamma(\nu)} \int_0^\tau x^{\nu - 1}(1 - x)^\mu\, dx.$$

But the integral in the expression above is simply the incomplete beta function. Thus

$$ {}_cD_t^{-\nu} t^{\mu} = \frac{t^{\mu+\nu}}{\Gamma(\nu)} \mathrm{B}_{\tau}(\nu, \mu + 1), \tag{3.39} $$

and if we let $c = 0$, formula (3.39) reduces to (3.1), as it should.

Furthermore, if we let $f(t)$ be e^{at} or $\cos at$ or $\sin at$, then (3.37) yields

$$ \begin{aligned} {}_cD_t^{-\nu} e^{at} &= e^{ac} E_{t-c}(\nu, a) \\ {}_cD_t^{-\nu} \cos at &= (\cos ac) C_{t-c}(\nu, a) - (\sin ac) S_{t-c}(\nu, a) \\ {}_cD_t^{-\nu} \sin at &= (\sin ac) C_{t-c}(\nu, a) + (\cos ac) S_{t-c}(\nu, a), \end{aligned} \tag{3.40} $$

which reduce to our previous formulas, (3.10), when $c = 0$. For a table of Riemann–Liouville fractional integrals, see [9] and Appendix D.

We conclude this section with a theoretical result. Suppose that f is continuous on $[0, X]$. Then the Riemann–Liouville fractional integral of f of order ν is

$$ \begin{aligned} D^{-\nu} f(t) &= \frac{1}{\Gamma(\nu)} \int_0^t (t - \xi)^{\nu-1} f(\xi)\, d\xi, \qquad \nu > 0, \quad 0 < t \leqq X \\ &= \frac{1}{\Gamma(\nu)} \int_0^t x^{\nu-1} f(t - x)\, dx. \end{aligned} \tag{3.41} $$

If, furthermore, we require that $f(x)$ be analytic at $x = a$ for all $a \in [0, X]$, the power series

$$ f(t - x) = f(t) + \sum_{k=1}^{\infty} (-1)^k \frac{D^k f(t)}{k!} x^k. \tag{3.42} $$

converges for all x in an interval that properly contains $[0, t]$. Thus it converges uniformly on $[0, t]$.

Now substitute (3.42) in (3.41),

$$ \begin{aligned} D^{-\nu} f(t) &= \frac{1}{\Gamma(\nu)} f(t) \int_0^t x^{\nu-1}\, dx \\ &\quad + \frac{1}{\Gamma(\nu)} \int_0^t x^{\nu} \left[\sum_{k=1}^{\infty} (-1)^k \frac{D^k f(t)}{k!} x^{k-1} \right] dx. \end{aligned} \tag{3.43} $$

By the uniform convergence we may interchange the order of summation and integration to obtain

$$D^{-\nu}f(t) = \frac{1}{\Gamma(\nu)} \sum_{k=0}^{\infty} \frac{(-1)^k D^k f(t)}{k!(\nu + k)} t^{\nu+k}, \qquad 0 \leqq t \leqq X. \quad (3.44)$$

Thus we have expressed the fractional integral of an analytic function in terms of ordinary derivatives of that function. If we recall that

$$D^{-\nu-k}(1) = \frac{1}{\Gamma(\nu + k + 1)} t^{\nu+k},$$

we also may write (3.44) as

$$\begin{aligned} D^{-\nu}f(t) &= \sum_{k=0}^{\infty} (-1)^k \binom{\nu + k - 1}{k} [D^k f(t)][D^{-\nu-k}(1)] \\ &= \sum_{k=0}^{\infty} \binom{-\nu}{k} [D^k f(t)][D^{-\nu-k}(1)] \end{aligned} \quad (3.45)$$

[see (B-2.6), p. 298].

4. DIRICHLET'S FORMULA

If $G(x, y)$ is jointly continuous on $[a, b] \times [a, b]$, we know from the elementary theory of functions that

$$\int_a^b dx \int_a^x G(x, y)\, dy = \int_a^b dy \int_y^b G(x, y)\, dx. \quad (4.1)$$

If, however, G is not continuous, but the integrals $\int_a^x G\, dy$ and $\int_y^b G\, dx$ exist as ordinary or improper Riemann integrals, general conditions under which the order of integration may be interchanged are difficult to obtain. Dirichlet's formula [48, p. 77] furnishes an example of a function for which (4.1) is true even though G may not be continuous. Because of the form of the integrand, this formula is well suited to the fractional calculus.

Dirichlet's Formula. Let F be jointly continuous on the Euclidean plane, and let λ, μ, ν be positive numbers. Then

$$\int_a^t (t-x)^{\mu-1}\,dx \int_a^x (y-a)^{\lambda-1}(x-y)^{\nu-1}F(x,y)\,dy$$
$$= \int_a^t (y-a)^{\lambda-1}\,dy \int_y^t (t-x)^{\mu-1}(x-y)^{\nu-1}F(x,y)\,dx. \quad (4.2)$$

Certain special cases are of particular interest. If $a = 0$, $\lambda = 1$, and $F(x, y) = g(x)f(y)$, then (4.2) becomes

$$\int_0^t (t-x)^{\mu-1}g(x)\,dx \int_0^x (x-y)^{\nu-1}f(y)\,dy$$
$$= \int_0^t f(y)\,dy \int_y^t (t-x)^{\mu-1}(x-y)^{\nu-1}g(x)\,dx. \quad (4.3)$$

Furthermore, if $g(x) \equiv 1$, (4.3) assumes the form

$$\int_0^t (t-x)^{\mu-1}\,dx \int_0^x (x-y)^{\nu-1}f(y)\,dy$$
$$= \mathrm{B}(\mu,\nu)\int_0^t (t-y)^{\mu+\nu-1}f(y)\,dy, \quad (4.4)$$

where B is the beta function.

As an important illustration of the usefulness of Dirichlet's formula, we shall prove the law of exponents for fractional integrals.

Theorem 1. Let f be continuous on J, and let $\mu, \nu > 0$. Then for all t,

$$D^{-\nu}[D^{-\mu}f(t)] = D^{-(\mu+\nu)}f(t) = D^{-\mu}[D^{-\nu}f(t)]. \quad (4.5)$$

Proof. By definition of the fractional integral,

$$D^{-\nu}[D^{-\mu}f(t)] = \frac{1}{\Gamma(\nu)}\int_0^t (t-x)^{\nu-1}\left[\frac{1}{\Gamma(\mu)}\int_0^x (x-y)^{\mu-1}f(y)\,dy\right]dx$$

and

$$D^{-(\mu+\nu)}f(t) = \frac{1}{\Gamma(\mu+\nu)}\int_0^t (t-y)^{\mu+\nu-1} f(y)\,dy.$$

Equation (4.4) now implies the truth of (4.5). ■

An alternative proof of this important theorem may be given by noting that

$$D^{-\nu}[D^{-\mu}P(t)] = D^{-(\mu+\nu)}P(t)$$

for any polynomial P, and then applying the Weierstrass approximation theorem, see [45].

If we wish Theorem 1 to be true when μ (or ν) is zero (which we do), we see that D^0 must be defined as the identity operator I. We shall make this identification.

For any positive integer p and continuous function f, we have seen that

$$D^{-p}f(t) = \frac{1}{(p-1)!}\int_0^t (t-x)^{p-1} f(x)\,dx \tag{4.6}$$

is the p-fold integral of $f(t)$. Thus if we let $\mu = p$ in (4.5), we have

$$D^{-p}[D^{-\nu}f(t)] = D^{-(p+\nu)}f(t) = D^{-\nu}[D^{-p}f(t)]. \tag{4.7}$$

We see, therefore, that the p-fold integral of the fractional integral $D^{-\nu}f(t)$ is the fractional integral of f of order $p+\nu$, and that they are both equal to the fractional integral of the p-fold integral of f of order ν.

As we have observed before, the fractional integral of an elementary function need not be elementary. We thus may use Theorem 1 to find the fractional integral of certain nonelementary functions. For example, if $f(t) = e^{at}$, then since e^{at} is continuous, Theorem 1 implies that

$$D^{-\nu}\left[D^{-\mu}e^{at}\right] = D^{-(\mu+\nu)}e^{at} \tag{4.8}$$

for positive μ and ν. But from (3.10), $D^{-\mu}e^{at} = E_t(\mu, a)$ and

$$D^{-(\mu+\nu)}e^{at} = E_t(\mu+\nu, a).$$

Thus with little effort we have established the formula

$$D^{-\nu}E_t(\mu, a) = E_t(\mu + \nu, a), \qquad \mu > -1, \quad \nu > 0 \tag{4.9}$$

[see (3.35)]. Similar arguments yield

$$D^{-\nu}C_t(\mu, a) = C_t(\mu + \nu, a), \qquad \mu > -1, \quad \nu > 0 \tag{4.10}$$

and

$$D^{-\nu}S_t(\mu, a) = S_t(\mu + \nu, a), \qquad \mu > -2, \quad \nu > 0. \tag{4.11}$$

Further formulas may be obtained by the use of (3.28) and (3.32) [or (3.33)]. For example, if we apply (3.28) to (4.9),

$$D^{-\nu}[tE_t(\mu, a)] = tE_t(\mu + \nu, a) - \nu E_t(\mu + \nu + 1, a), \qquad \mu > -2, \quad \nu > 0. \tag{4.12}$$

5. DERIVATIVES OF THE FRACTIONAL INTEGRAL AND THE FRACTIONAL INTEGRAL OF DERIVATIVES

In Section III-4 we showed that the integral of the fractional integral was the fractional integral of the integral. We now develop similar formulas involving derivatives. Unfortunately, the relations are not quite as simple. The basic rules for manipulating these quantities are given below in Theorem 2. Some examples of $D^p[D^{-\nu}f(t)]$ and $D^{-\nu}[D^p f(t)]$ (where p is a positive integer) will be given.

Theorem 2. Let f be continuous on J and let $\nu > 0$. Then:
(a) If Df is of class **C**, then

$$D^{-\nu-1}[Df(t)] = D^{-\nu}f(t) - \frac{f(0)}{\Gamma(\nu + 1)}t^{\nu}$$

and
(b) If Df is continuous on J, then for $t > 0$,

$$D[D^{-\nu}f(t)] = D^{-\nu}[Df(t)] + \frac{f(0)}{\Gamma(\nu)}t^{\nu-1}.$$

Proof. To prove part (a), let $\epsilon > 0$, $\eta > 0$ be assigned. Then $(t - \xi)^{\nu - 1}$ and $f(\xi)$ are continuously differentiable on $[\eta, t - \epsilon]$. Thus an integration by parts establishes

$$\int_{\eta}^{t-\epsilon} (t - \xi)^{\nu} [Df(\xi)] \, d\xi = \nu \int_{\eta}^{t-\epsilon} (t - \xi)^{\nu - 1} f(\xi) \, d\xi + \epsilon^{\nu} f(t - \epsilon) - (t - \eta)^{\nu} f(\eta).$$

Now take the limit as ϵ and η independently approach zero and divide by $\Gamma(\nu + 1)$ to obtain part (a).

To prove part (b), make the change of variable

$$\xi = t - x^{\lambda} \tag{5.1}$$

(where $\lambda = 1/\nu$) in

$$D^{-\nu} f(t) = \frac{1}{\Gamma(\nu)} \int_0^t (t - \xi)^{\nu - 1} f(\xi) \, d\xi$$

to obtain

$$D^{-\nu} f(t) = \frac{1}{\Gamma(\nu + 1)} \int_0^{t^{\nu}} f(t - x^{\lambda}) \, dx.$$

Then for $t > 0$,

$$D[D^{-\nu} f(t)] = \frac{1}{\Gamma(\nu + 1)} \left[f(0)(\nu t^{\nu - 1}) + \int_0^{t^{\nu}} \frac{\partial}{\partial t} f(t - x^{\lambda}) \, dx \right].$$

Now reversing the transformation (5.1), that is, letting $t - x^{\lambda} = \xi$, proves part (b). ■

If we apply Theorem 2 to the special case

$$f(t) = t^{\mu}, \qquad \mu > 0,$$

then both parts (a) and (b) yield identities.

Now let $f(t) = e^{at}$. Then part (a) implies that

$$D^{-\nu - 1}[ae^{at}] = D^{-\nu}[e^{at}] - \frac{t^{\nu}}{\Gamma(\nu + 1)},$$

and using (3.10),

$$aE_t(\nu + 1, a) = E_t(\nu, a) - \frac{t^\nu}{\Gamma(\nu + 1)}, \tag{5.2}$$

a recursion formula for the E_t function that may be found in Appendix C. If we apply part (b) to e^{at}, then

$$DE_t(\nu, a) = aE_t(\nu, a) + \frac{t^{\nu-1}}{\Gamma(\nu)}$$

and using (5.2) we see that

$$DE_t(\nu, a) = E_t(\nu - 1, a), \tag{5.3}$$

a differentiation formula for E_t that also may be found in Appendix C. Thus we see that an application of Theorem 2 results in a painless derivation of such formulas as (5.2) and (5.3).

If $f(t) = \cos at$, then using (3.10), p. 49, we see that parts (a) and (b) of Theorem 2 yield

$$-aS_t(\nu + 1, a) = C_t(\nu, a) - \frac{t^\nu}{\Gamma(\nu + 1)} \tag{5.4}$$

and

$$DC_t(\nu, a) = -aS_t(\nu, a) + \frac{t^{\nu-1}}{\Gamma(\nu)},$$

respectively. Replacing ν by $\nu - 1$ in (5.4) and substituting in the equation above yields the differentiation formula

$$DC_t(\nu, a) = C_t(\nu - 1, a). \tag{5.5}$$

Similarly, we see that if we apply Theorem 2 to $f(t) = \sin at$, we obtain

$$aC_t(\nu + 1, a) = S_t(\nu, a) \tag{5.6}$$

and

$$DS_t(\nu, a) = S_t(\nu - 1, a). \tag{5.7}$$

Formulas (5.4), (5.5), (5.6), and (5.7) also may be found in Appendix C.

Using (5.4), we may write (3.15) in the neat form

$$D^{-\nu}\sin^2 at = aS_t(\nu+1,2a). \tag{5.8}$$

We may generalize Theorem 2 to derivatives of higher order.

Theorem 3. Let p be a positive integer. Let $D^{p-1}f$ be continuous on J, and let $\nu > 0$. Then:

(a) If $D^p f$ is of class **C**, then

$$D^{-\nu}f(t) = D^{-\nu-p}[D^p f(t)] + Q_p(t,\nu)$$

and

(b) if $D^p f$ is continuous on J, then for $t > 0$

$$D^p[D^{-\nu}f(t)] = D^{-\nu}[D^p f(t)] + Q_p(t,\nu-p),$$

where

$$Q_p(t,\nu) = \sum_{k=0}^{p-1}\frac{t^{\nu+k}}{\Gamma(\nu+k+1)}D^k f(0). \tag{5.9}$$

Proof. Replacing ν by $\nu+1$ and f by Df in part (a) of Theorem 2 yields

$$D^{-\nu-2}[D^2 f(t)] = D^{-\nu-1}[Df(t)] - \frac{Df(0)}{\Gamma(\nu+2)}t^{\nu+1}.$$

Now replace $D^{-\nu-1}[Df(t)]$ in the expression above by part (a) of Theorem 2 to obtain

$$D^{-\nu-2}[D^2 f(t)] = D^{-\nu}f(t) - \frac{f(0)}{\Gamma(\nu+1)}t^{\nu} - \frac{Df(0)}{\Gamma(\nu+2)}t^{\nu+1}.$$

Repeated iterations establish part (a).

To prove part (b), differentiate part (b) of Theorem 2 to obtain (for $t > 0$)

$$D^2[D^{-\nu}f(t)] = D\{D^{-\nu}[Df(t)]\} + \frac{f(0)}{\Gamma(\nu-1)}t^{\nu-2}.$$

Now the term in braces is given by part (b) of Theorem 2 with f

replaced by Df. Hence

$$D^2[D^{-\nu}f(t)] = D^{-\nu}[D^2 f(t)] + \frac{f(0)}{\Gamma(\nu-1)}t^{\nu-2} + \frac{Df(0)}{\Gamma(\nu)}t^{\nu-1}.$$

Repeated iterations establish part (b). ■

Since $Q_p(t,\nu)$ may be expressed as a fractional integral, that is,

$$Q_p(t,\nu) = D^{-\nu}[R_p(t)], \tag{5.10}$$

where

$$R_p(t) = \sum_{k=0}^{p-1} \frac{D^k f(0)}{k!} t^k, \tag{5.11}$$

we may write part (a) of Theorem 3 as

$$D^{-\nu}[f(t) - R_p(t)] = D^{-\nu-p}[D^p f(t)]. \tag{5.12}$$

As a corollary to Theorem 3, we see that if $D^k f(0) = 0$, $k = 0, 1, \ldots, p-1$, then

$$D^{-\nu}f(t) = D^{-\nu-p}[D^p f(t)] \tag{5.13}$$

and

$$D^p[D^{-\nu}f(t)] = D^{-\nu}[D^p f(t)]. \tag{5.14}$$

These formulas are generalized in Chapter IV.

Before continuing our theoretical development, let us consider some consequences of Theorem 3. If we apply part (a) to the function $f(t) = e^{at}$, then

$$D^{-\nu}[e^{at}] = a^p D^{-\nu-p}[e^{at}] + Q_p(t,\nu), \tag{5.15}$$

where

$$Q_p(t,\nu) = \sum_{k=0}^{p-1} a^k \frac{t^{\nu+k}}{\Gamma(\nu+k+1)}.$$

Thus from (3.10) we see that (5.15) reduces to the recursion formula

$$E_t(\nu,a) = a^p E_t(\nu+p,a) + \sum_{k=0}^{p-1} a^k \frac{t^{\nu+k}}{\Gamma(\nu+k+1)} \tag{5.16}$$

[see (C-3.4), p. 315]. On the other hand, part (b) implies that

$$D^p E_t(\nu, a) = a^p E_t(\nu, a) + \sum_{k=0}^{p-1} a^k \frac{t^{\nu+k-p}}{\Gamma(\nu + k + 1 - p)} \tag{5.17}$$

see (C-3.5), p. 316]. If we replace ν by $\nu - p$ in (5.16) and substitute in (5.17), we have the elegant formula

$$D^p E_t(\nu, a) = E_t(\nu - p, a), \qquad p = 0, 1, \ldots \tag{5.18}$$

[which also could have been obtained by iterating (5.3)].

Similar arguments, of course, establish that

$$C_t(\nu, a) = (-1)^{p/2} a^p C_t(\nu + p, a) + \sum_{j=0}^{(1/2)p-1} (-1)^j a^{2j} \frac{t^{\nu+2j}}{\Gamma(\nu + 2j + 1)} \tag{5.19}$$

if p is even, and

$$C_t(\nu, a) = (-1)^{(1/2)(p+1)} a^{p+1} C_t(\nu + p + 1, a) + \sum_{j=0}^{(1/2)(p-1)} (-1)^j a^{2j} \frac{t^{\nu+2j}}{\Gamma(\nu + 2j + 1)} \tag{5.20}$$

if p is odd, and

$$S_t(\nu, a) = (-1)^{p/2} a^p S_t(\nu + p, a) + \sum_{j=0}^{(1/2)p-1} (-1)^j a^{2j+1} \frac{t^{\nu+2j+1}}{\Gamma(\nu + 2j + 2)} \tag{5.21}$$

if p is even, and

$$S_t(\nu, a) = (-1)^{(1/2)(p+1)} a^{p+1} S_t(\nu + p + 1, a) + \sum_{j=0}^{(1/2)(p-1)} (-1)^j a^{2j+1} \frac{t^{\nu+2j+1}}{\Gamma(\nu + 2j + 2)} \tag{5.22}$$

if p is odd; while

$$D^p C_t(\nu, a) = C_t(\nu - p, a) \tag{5.23}$$

and

$$D^p S_t(\nu, a) = S_t(\nu - p, a) \tag{5.24}$$

for $p = 0, 1, \ldots$.

In the spirit of Theorems 2 and 3 and (5.13) and (5.14) we shall prove a theorem that expresses the derivative of a fractional integral of a function as a fractional integral of that function.

Theorem 4. Let f have a continuous derivative on J. Let p be a positive integer and let $\nu > p$. Then for all $t \in J$,

$$D^p[D^{-\nu}f(t)] = D^{-(\nu-p)}f(t). \tag{5.25}$$

Proof. By Definition 1,

$$D^{-\nu}f(t) = \frac{1}{\Gamma(\nu)}\int_0^t (t-\xi)^{\nu-1}f(\xi)\,d\xi$$

and

$$D^{p-1}[D^{-\nu}f(t)] = D^{-(\nu-p)-1}f(t) \tag{5.26}$$

since $\nu > p$. Differentiation of the expression above leads to

$$D^p[D^{-\nu}f(t)] = D[D^{p-1-\nu}f(t)].$$

If we replace ν by $\nu - p + 1$ in part (b) of Theorem 2, and then substitute this result for the right-hand side of the formula above, we get

$$D^p[D^{-\nu}f(t)] = D^{p-1-\nu}[Df(t)] + \frac{f(0)}{\Gamma(\nu+1-p)}t^{\nu-p}. \tag{5.27}$$

Now replace ν by $\nu - p$ in part (a) of Theorem 2 and substitute in (5.27). ■

Suppose that q is a positive integer and let $\mu > q$. Then from Theorem 4,

$$D^q[D^{-\mu}f(t)] = D^{-(\mu-q)}f(t). \tag{5.28}$$

Suppose further that

$$p - \nu = q - \mu. \tag{5.29}$$

Then we have the interesting corollary that

$$D^p[D^{-\nu}f(t)] = D^q[D^{-\mu}f(t)]. \tag{5.30}$$

In the next theorem we generalize this result by showing that (5.30) is true even if $p > \nu$ and $q > \mu$, and also exhibit the relation between $D^{-\nu}[D^p f(t)]$ and $D^{-\mu}[D^q f(t)]$.

Theorem 5. Let p and q be positive integers and let μ and ν be positive numbers such that

$$p - \nu = q - \mu. \tag{5.31}$$

Let f have r continuous derivatives on J where $r = \max(p, q)$. Then for all $t \in J$,

$$D^{-\nu}[D^p f(t)] = D^{-\mu}[D^q f(t)] + \operatorname{sgn}(q - p) \sum_{k=s}^{r-1} \frac{t^{\nu - p + k}}{\Gamma(\nu - p + k + 1)} D^k f(0), \tag{5.32}$$

where $s = \min(p, q)$, and for all $t \in J'$,

$$D^q[D^{-\mu}f(t)] = D^p[D^{-\nu}f(t)]. \tag{5.33}$$

Proof. If $p = q$, the theorem is trivial. Suppose then that $q > p$. Let $\sigma = q - p > 0$. Then from (5.31) we have

$$\mu = \nu + \sigma > 0.$$

From part (a) of Theorem 3

$$D^{-\nu}[D^p f(t)] = D^{-\nu-\sigma}[D^{\sigma+p} f(t)] + \sum_{k=0}^{\sigma-1} \frac{t^{\nu+k}}{\Gamma(\nu + k + 1)} D^{k+p} f(0).$$

Now recall that $\nu + \sigma = \mu$ and $\sigma + p = q$. Thus we have proved (5.32).

To prove (5.33) we have from Theorem 4 that

$$D^{\sigma}[D^{-\nu-\sigma}f(t)] = D^{-\nu}f(t). \tag{5.34}$$

If we differentiate (5.34) p times,

$$D^{p+\sigma}[D^{-\nu-\sigma}f(t)] = D^{p}[D^{-\nu}f(t)].$$

But $p + \sigma = q$ and $\nu + \sigma = \mu$. Thus we have established (5.33). ■

6. LAPLACE TRANSFORM OF THE FRACTIONAL INTEGRAL

The Laplace transform will prove to be an indispensable tool, especially in our study of fractional differential equations. We briefly inaugurate our discussion of this powerful method in the present section. In future chapters as well as in Appendix C we consider additional information about, and applications of, this important technique.

We recall that a function $f(t)$ defined on J' is said to be of exponential order α if there exist positive constants M and T such that

$$e^{-\alpha t}|f(t)| \leqq M$$

for all $t \geqq T$. If $f(t)$ is of class **C** and of exponential order α, then

$$\int_0^{\infty} f(t)e^{-st}\,dt \tag{6.1}$$

exists for all s with $\operatorname{Re} s > \alpha$. We shall call (6.1) the *Laplace transform* of $f(t)$ and write

$$\mathscr{L}\{f(t)\} = \int_0^{\infty} f(t)e^{-st}\,dt.$$

Sometimes it is convenient to denote the Laplace transform of f by F,

$$F(s) = \mathscr{L}\{f(t)\}.$$

We shall also have occasion to write

$$f(t) = \mathscr{L}^{-1}\{F(s)\}$$

to indicate that f is the (unique) inverse Laplace transform of F.

If f and g are of exponential order, then clearly $f(t)g(t)$ is of exponential order. We also assert that if f is continuous on J and Df is of class **C**, then if Df is of exponential order, so is f. To demonstrate this we first note that if $\epsilon > 0$, then

$$\int_\epsilon^t [Df(\xi)]\, d\xi = f(t) - f(\epsilon),$$

and since f is continuous on J,

$$f(t) = f(0) + \int_0^t [Df(\xi)]\, d\xi.$$

By hypothesis Df is of exponential order. Hence there exists an α (which we shall assume to be positive) and constants T and M such that

$$|e^{-\alpha t} Df(t)| < M \tag{6.2}$$

for all $t > T$. Now if we write

$$f(t) = f(0) + \int_0^t e^{\alpha\xi}[e^{-\alpha\xi} Df(\xi)]\, d\xi$$

(i.e., we have multiplied the integrand by $1 = e^{\alpha\xi}e^{-\alpha\xi}$), then

$$f(t) = f(0) + \int_0^T Df(\xi)\, d\xi + \int_T^t e^{\alpha\xi}[e^{-\alpha\xi} Df(\xi)]\, d\xi, \qquad t > T,$$

and by (6.2),

$$|f(t)| \leqq A + M\int_T^t e^{\alpha\xi}\, d\xi,$$

where A is a positive constant. But

$$\int_T^t e^{\alpha\xi}\, d\xi < \frac{e^{\alpha t}}{\alpha}.$$

Thus for all $t \geqq T$,

$$|f(t)| < M'e^{\alpha t}$$

for some M'. Hence $f(t)$ is of exponential order.

If a function of class **C** has compact support, then the condition that f be of exponential order is vacuous.

The functions t^{μ} $(\mu > -1)$, e^{at}, $t^{\mu-1}e^{at}$ $(\mu > 0)$, $\cos at$, and $\sin at$ all are of class **C** and of exponential order. Some elementary calculus then shows that

$$\mathscr{L}\{t^{\mu}\} = \frac{\Gamma(\mu+1)}{s^{\mu+1}}, \qquad \mu > -1 \tag{6.3a}$$

$$\mathscr{L}\{e^{at}\} = \frac{1}{s-a} \tag{6.3b}$$

$$\mathscr{L}\{t^{\mu-1}e^{at}\} = \frac{\Gamma(\mu)}{(s-a)^{\mu}}, \qquad \mu > 0 \tag{6.3c}$$

$$\mathscr{L}\{\cos at\} = \frac{s}{s^2+a^2} \tag{6.3d}$$

$$\mathscr{L}\{\sin at\} = \frac{a}{s^2+a^2}. \tag{6.3e}$$

One of the most useful properties of the Laplace transform is embodied in the *convolution* theorem (see [7]). The theorem states that the Laplace transform of the convolution of two functions is the product of their Laplace transforms. Thus if $F(s)$ and $G(s)$ are the Laplace transforms of $f(t)$ and $g(t)$, respectively, then

$$\mathscr{L}\left\{\int_0^t f(t-\xi)g(\xi)\,d\xi\right\} = F(s)G(s). \tag{6.4}$$

Now if f is of class **C**, the fractional integral of f of order ν is

$$D^{-\nu}f(t) = \frac{1}{\Gamma(\nu)}\int_0^t (t-\xi)^{\nu-1}f(\xi)\,d\xi, \qquad \nu > 0,$$

which is a convolution integral. Thus if f is of exponential order

$$\mathscr{L}\{D^{-\nu}f(t)\} = \frac{1}{\Gamma(\nu)}\mathscr{L}\{t^{\nu-1}\}\mathscr{L}\{f(t)\} \tag{6.5a}$$

$$= s^{-\nu}F(s), \qquad \nu > 0, \tag{6.5b}$$

where F is the Laplace transform of f. We observe that (6.5b) is valid even if $\nu = 0$, but that (6.5a) is indeterminate. However,

$$\lim_{\nu \to 0} \mathscr{L}\left\{\frac{t^{\nu-1}}{\Gamma(\nu)}\right\} = 1. \tag{6.6}$$

As examples of (6.5) we see from (6.3) that

$$\mathscr{L}\{D^{-\nu}t^{\mu}\} = \frac{\Gamma(\mu+1)}{s^{\mu+\nu+1}}, \qquad \nu > 0, \quad \mu > -1 \tag{6.7a}$$

$$\mathscr{L}\{D^{-\nu}e^{at}\} = \frac{1}{s^{\nu}(s-a)}, \qquad \nu > 0 \tag{6.7b}$$

$$\mathscr{L}\{D^{-\nu}t^{\mu-1}e^{at}\} = \frac{\Gamma(\mu)}{s^{\nu}(s-a)^{\mu}}, \qquad \nu > 0, \quad \mu > 0 \tag{6.7c}$$

$$\mathscr{L}\{D^{-\nu}\cos at\} = \frac{1}{s^{\nu-1}(s^2+a^2)}, \qquad \nu > 0 \tag{6.7d}$$

$$\mathscr{L}\{D^{-\nu}\sin at\} = \frac{a}{s^{\nu}(s^2+a^2)}, \qquad \nu > 0. \tag{6.7e}$$

We turn now to the problem of finding the Laplace transform of the fractional integral of the derivative and the Laplace transform of the derivative of the fractional integral. Suppose then that f is continuous on J and Df is of class **C** and of exponential order. Then, by (6.5),

$$\begin{aligned}\mathscr{L}\{D^{-\nu}[Df(t)]\} &= s^{-\nu}\mathscr{L}\{Df(t)\}\\ &= s^{-\nu}[sF(s) - f(0)], \qquad \nu > 0,\end{aligned} \tag{6.8}$$

where $F(s)$ is the Laplace transform of $f(t)$. Since $f(t)$ is continuous on J by hypothesis, $f(0)$ exists. Thus we have found the Laplace transform of the fractional integral of the derivative. This formula is obviously also valid if $\nu = 0$.

Now we consider the problem of finding the Laplace transform of the derivative of the fractional integral. From part (b) of Theorem 2, p. 60,

$$\begin{aligned}\mathscr{L}\{D[D^{-\nu}f(t)]\} &= \mathscr{L}\{D^{-\nu}[Df(t)]\} + f(0)\mathscr{L}\left\{\frac{t^{\nu-1}}{\Gamma(\nu)}\right\}\\ &= s^{-\nu}[sF(s) - f(0)] + s^{-\nu}f(0)\\ &= s^{1-\nu}F(s), \qquad \nu > 0\end{aligned} \tag{6.9}$$

[where we have used (6.8)]. Now we know that if $\nu = 0$,

$$\mathscr{L}\{Df(t)\} = sF(s) - f(0). \tag{6.10}$$

But this is not the same result we would get if we let $\nu = 0$ in (6.9). This "discontinuity" arises from the fact that

$$\lim_{\nu \to 0} \frac{t^{\nu-1}}{\Gamma(\nu)} = 0, \tag{6.11}$$

and comparing with (6.6) we see that "$\mathscr{L}$" and "lim" do not commute.

Returning to (6.7) and recalling (3.10), we see that

$$\begin{aligned}
\mathscr{L}\{E_t(\nu, a)\} &= \frac{1}{s^\nu(s-a)}, \qquad \nu > 0 \\
\mathscr{L}\{C_t(\nu, a)\} &= \frac{1}{s^{\nu-1}(s^2+a^2)}, \qquad \nu > 0 \\
\mathscr{L}\{S_t(\nu, a)\} &= \frac{a}{s^\nu(s^2+a^2)}, \qquad \nu > 0.
\end{aligned} \tag{6.12}$$

We elaborate on these formulas in Section C-4. Thus we see that with the aid of the fractional calculus, we have found, with little effort, the Laplace transforms of some nonelementary functions.

For completeness, from (6.7c),

$$\mathscr{L}^{-1}\left\{\frac{\Gamma(\mu)}{s^\nu(s-a)^\mu}\right\} = \frac{1}{\Gamma(\nu)}\int_0^t (t-\xi)^{\nu-1}\xi^{\mu-1}e^{a\xi}\,d\xi.$$

Thus from (B-4.8), p. 305, we have

$$D^{-\nu}\left[t^{\mu-1}e^{at}\right] = \frac{\Gamma(\mu)}{\Gamma(\mu+\nu)}t^{\mu+\nu-1}\,{}_1F_1(\mu, \mu+\nu; at),$$

$$\nu > 0, \quad \mu > 0. \tag{6.13}$$

[If μ is a positive integer, see (3.34), p. 54 and (C-4.5), p. 323.]

Finally, we wish to mention a phenomenon that some readers might not have noticed. Although this phenomenon is prevalent in all of mathematics, we wish to emphasize it in our dealings with the Laplace transform. Depending on the method used, it is sometimes possible to weaken a set of hypotheses and still arrive at the same conclusion. For example, if

$$F(s) = \frac{1}{s}$$

and

$$G(s) = \frac{1}{s^\nu}$$

then $G(s)$ is meaningful only if $\nu > 0$, but $F(s)G(s)$ is meaningful if $\nu > -1$. Thus if we find the inverse transform of $F(s)G(s)$ directly, namely

$$\mathscr{L}^{-1}\left\{\frac{1}{s^{1+\nu}}\right\} = \frac{t^\nu}{\Gamma(\nu+1)}, \tag{6.14}$$

we need require only the weaker hypothesis, $\nu > -1$. But if we use the convolution approach, namely,

$$\int_0^t \frac{(t-\xi)^{\nu-1}}{\Gamma(\nu)} \xi^0 \, d\xi = \frac{t^\nu}{\Gamma(\nu+1)}, \tag{6.15}$$

then since the integral is meaningful only for $\nu > 0$, we have proved our result only with the stronger hypothesis $\nu > 0$ (even though we know that the result is valid for $\nu > -1$).

As another more subtle example, let

$$x(t) = t^{\lambda-1}$$

and as our problem let it be required to find the inverse Laplace transform of

$$Y(s) = \frac{s^2 X(s)}{s^2+1}.$$

Now

$$X(s) = \frac{\Gamma(\lambda)}{s^\lambda} \tag{6.16}$$

provided that $\lambda > 0$ and

$$Y(s) = \frac{\Gamma(\lambda)}{s^{\lambda-2}(s^2+1)}$$

is meaningful if $\lambda > 0$. Thus

$$y(t) = \Gamma(\lambda) S_t(\lambda - 2, 1), \qquad \lambda > 0. \tag{6.17}$$

On the other hand, if we write

$$s^2 X(s) = \mathscr{L}\{D^2 x(t)\} + sx(0) + Dx(0), \tag{6.18}$$

then $Y(s)$ may be expressed as

$$Y(s) = \frac{\mathscr{L}\{D^2 x(t)\}}{s^2+1} + \frac{sx(0) + Dx(0)}{x^2+1}.$$

But from (6.16),

$$s^2 X(s) = \frac{\Gamma(\lambda)}{s^{\lambda-2}},$$

which is meaningful only for $\lambda > 2$. If this is the case, $x(0) = 0 = Dx(0)$ and by the convolution theorem

$$\begin{aligned} y(t) &= \int_0^t \sin(t-\xi) D^2 x(\xi)\, d\xi \\ &= (\lambda-1)(\lambda-2)\int_0^t \sin(t-\xi)\xi^{\lambda-3}\, d\xi \\ &= \Gamma(\lambda) S_t(\lambda-2, 1) \end{aligned}$$

[see (C-3.20), p. 320]. Thus we have proved our result only for $\lambda > 2$, while we know from (6.17) that it is valid for $\lambda > 0$.

7. LEIBNIZ'S FORMULA FOR FRACTIONAL INTEGRALS

A Leibniz-type formula expresses the result of operating on the product of two functions as a sum of products of operations performed on each function. The classical Leibniz rule or formula of elementary calculus is

$$D^n[f(t)g(t)] = \sum_{k=0}^{n} \binom{n}{k} [D^k g(t)][D^{n-k} f(t)], \tag{7.1}$$

where f and g are assumed to be n-fold differentiable on some interval. Now we wish to extend (7.1) to fractional operators.

We have seen in Section III-3 that if f is of class **C** and $g(t) = t^p$, where p is a positive integer, then the fractional integral of the product fg of order $\nu > 0$ may be written as

$$D^{-\nu}[f(t)g(t)] = \sum_{k=0}^{p} \binom{-\nu}{k} [D^k g(t)][D^{-\nu-k} f(t)] \tag{7.2}$$

[see (3.33), p. 53]. The resemblance of this formula to (7.1) is obvious. The immediate problem we wish to address is the extension of (7.2) to the case where g is not just a simple polynomial. Later, in Chapter IV, we extend these formulas to fractional derivatives.

Suppose then that f is continuous on $[0, X]$ and that g is analytic at a for all $a \in [0, X]$. Then fg is certainly of class **C**, and for $\nu > 0$, the fractional integral

$$D^{-\nu}[f(t)g(t)] = \frac{1}{\Gamma(\nu)} \int_0^t (t-\xi)^{\nu-1} [f(\xi)g(\xi)]\, d\xi, \qquad 0 < t \leqq X \tag{7.3}$$

exists. We may write

$$\begin{aligned} g(\xi) &= \sum_{k=0}^{\infty} (-1)^k \frac{D^k g(t)}{k!} (t-\xi)^k \\ &= g(t) + \sum_{k=1}^{\infty} (-1)^k \frac{D^k g(t)}{k!} (t-\xi)^k. \end{aligned} \tag{7.4}$$

The series (7.4) converges for all ξ in an interval that properly contains $[0, t]$, and hence uniformly on $[0, t]$.

Now substitute (7.4) into (7.3) to obtain

$$\begin{aligned} D^{-\nu}[f(t)g(t)] &= g(t) D^{-\nu} f(t) + \frac{1}{\Gamma(\nu)} \int_0^t (t-\xi)^{\nu} f(\xi) \\ &\quad \times \sum_{k=1}^{\infty} (-1)^k \frac{D^k g(t)}{k!} (t-\xi)^{k-1}\, d\xi. \end{aligned} \tag{7.5}$$

Since f is continuous on $[0, X]$ and $\nu > 0$,

$$(t - \xi)^{\nu} f(\xi)$$

is bounded on $[0, t]$. Hence we may interchange the order of integration and summation in (7.5) to obtain

$$\begin{aligned} D^{-\nu}[f(t)g(t)] &= \sum_{k=0}^{\infty} (-1)^k \frac{\Gamma(\nu + k)}{k!\,\Gamma(\nu)} [D^k g(t)][D^{-\nu-k} f(t)] \\ &= \sum_{k=0}^{\infty} \binom{-\nu}{k} [D^k g(t)][D^{-\nu-k} f(t)] \end{aligned} \tag{7.6}$$

[see (B-2.6), p. 298].

Thus we have shown:

Theorem 6. Let f be continuous on $[0, X]$, and let g be analytic at a for all $a \in [0, X]$. Then for $\nu > 0$ and $0 < t \leqq X$,

$$D^{-\nu}[f(t)g(t)] = \sum_{k=0}^{\infty} \binom{-\nu}{k} [D^k g(t)][D^{-\nu-k} f(t)]. \tag{7.7}$$

We call (7.7) the Leibniz formula for fractional integrals. Equation (7.2) is a special case.

Note: The only reason we assumed g analytic for all points a in $[0, X]$ was to guarantee the uniform convergence of (7.4) for $\xi \in [0, t]$.

As our first application of the Leibniz rule, let $f(t) = t^{\lambda}$, $\lambda \geqq 0$, and let $g(t) = e^t$. Then from Theorem 6

$$\begin{aligned} D^{-\nu}[t^{\lambda} e^t] &= \sum_{k=0}^{\infty} (-1)^k \frac{\Gamma(\nu + k)}{k!\,\Gamma(\nu)} [e^t] \left[\frac{\Gamma(\lambda + 1)}{\Gamma(\lambda + \nu + k + 1)} t^{\lambda+\nu+k} \right] \\ &= \frac{\Gamma(\lambda + 1)}{\Gamma(\lambda + \nu + 1)} t^{\lambda+\nu} e^t {}_1F_1(\nu, \lambda + \nu + 1; -t). \end{aligned} \tag{7.8}$$

Using this result we may deduce a useful identity involving the confluent hypergeometric functions [see (7.11)]. For, from the definition of the fractional integral,

$$\begin{aligned} D^{-\nu}[t^{\lambda} e^t] &= \frac{1}{\Gamma(\nu)} \int_0^t (t - \xi)^{\nu-1} \xi^{\lambda} e^{\xi}\, d\xi \\ &= \frac{\Gamma(\lambda + 1)}{\Gamma(\lambda + \nu + 1)} t^{\lambda+\nu} {}_1F_1(\lambda + 1, \lambda + \nu + 1; t) \end{aligned} \tag{7.9}$$

by (6.13) [see also (3.34), p. 54]. Comparing (7.8) and (7.9) establishes the identity

$$e^t {}_1F_1(\nu, \lambda + \nu + 1; -t) = {}_1F_1(\lambda + 1, \lambda + \nu + 1; t). \quad (7.10)$$

Or if we let

$$a = \lambda + 1$$
$$c = \lambda + \nu + 1,$$

we have, in more conventional notation,

$${}_1F_1(a, c; t) = e^t {}_1F_1(c - a, c; -t) \quad (7.11)$$

[see (B-4.10), p. 305].

As a second example, let $f(t) = t^\lambda$, $\lambda \geqq 0$, and let $g(t) = (1 - t)^{-\alpha}$. Let X be a fixed positive number less than 1. Then $(1 - t)^{-\alpha}$ is analytic at every point of $[0, X]$, and by Theorem 6,

$$D^{-\nu}\left[t^\lambda(1 - t)^{-\alpha}\right] = \sum_{k=0}^{\infty} (-1)^k \frac{\Gamma(\nu + k)}{k!\,\Gamma(\nu)}\left[D^k(1 - t)^{-\alpha}\right]\left[D^{-\nu-k}t^\lambda\right]$$

for $0 < t \leqq X$. But

$$D^k(1 - t)^{-\alpha} = \frac{\Gamma(\alpha + k)}{\Gamma(\alpha)}(1 - t)^{-\alpha-k}$$

and

$$D^{-\nu-k}t^\lambda = \frac{\Gamma(\lambda + 1)}{\Gamma(\lambda + \nu + k + 1)}t^{\lambda+\nu+k}.$$

Thus

$$\begin{aligned} D^{-\nu}\left[t^\lambda(1 - t)^{-\alpha}\right] &= \frac{\Gamma(\lambda + 1)}{\Gamma(\nu)\Gamma(\alpha)}t^{\lambda+\nu}(1 - t)^{-\alpha} \\ &\quad \times \sum_{k=0}^{\infty} \frac{\Gamma(\nu + k)\Gamma(\alpha + k)}{\Gamma(\lambda + \nu + k + 1)k!}\left(\frac{t}{t - 1}\right)^k \\ &= \frac{\Gamma(\lambda + 1)}{\Gamma(\lambda + \nu + 1)}t^{\lambda+\nu}(1 - t)^{-\alpha} \\ &\quad \times {}_2F_1\left(\nu, \alpha, \lambda + \nu + 1; \frac{t}{t - 1}\right). \end{aligned} \quad (7.12)$$

On the other hand, since

$$(1-t)^{-\alpha} = {}_1F_0(\alpha;t) = \frac{1}{\Gamma(\alpha)}\sum_{k=0}^{\infty}\frac{\Gamma(\alpha+k)}{k!}t^k$$

for $|t| < 1$ [see (B-4.13), p. 305],

$$\begin{aligned} D^{-\nu}\left[t^{\lambda}(1-t)^{-\alpha}\right] &= \frac{1}{\Gamma(\alpha)}D^{-\nu}\left[\sum_{k=0}^{\infty}\frac{\Gamma(\alpha+k)}{k!}t^{k+\lambda}\right] \\ &= \frac{1}{\Gamma(\nu)\Gamma(\alpha)}\int_0^t (t-\xi)^{\nu-1}\sum_{k=0}^{\infty}\frac{\Gamma(\alpha+k)}{k!}\xi^{k+\lambda}\,d\xi \\ &= \frac{\Gamma(\lambda+1)}{\Gamma(\lambda+\nu+1)}t^{\lambda+\nu}{}_2F_1(\lambda+1,\alpha,\lambda+\nu+1;t). \end{aligned} \tag{7.13}$$

Comparing (7.12) and (7.13) leads to

$$(1-t)^{-\alpha}{}_2F_1\left(\nu,\alpha,\lambda+\nu+1;\frac{t}{t-1}\right) = {}_2F_1(\lambda+1,\alpha,\lambda+\nu+1;t).$$

Or in more conventional notation with

$$\begin{aligned} a &= \lambda+1 \\ b &= \alpha \\ c &= \lambda+\nu+1, \end{aligned}$$

we have established the identity

$$(1-t)^{-b}{}_2F_1\left(c-a,b,c;\frac{t}{t-1}\right) = {}_2F_1(a,b,c;t) \tag{7.14}$$

between hypergeometric functions [see (B-4.6), p. 304].

Another interesting result that we may establish using the Leibniz rule is the identity

$${}_2F_1(a,b,c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \tag{7.15}$$

[sometimes called Laurent's formula; see (B-4.4), p. 304].

To prove (7.15) we start with the trivial identity

$$t^{\lambda+\mu} = t^{\lambda} t^{\mu}, \qquad t > 0. \tag{7.16}$$

Now for $\nu > 0$ and $\lambda + \mu > -1$,

$$D^{-\nu} t^{\lambda+\mu} = \frac{\Gamma(\lambda + \mu + 1)}{\Gamma(\lambda + \mu + \nu + 1)} t^{\lambda+\mu+\nu}. \tag{7.17}$$

We shall show that if $\lambda, \mu \geqq 0$, we may apply Leibniz's formula to the product of $f(t) = t^{\lambda}$ and $g(t) = t^{\mu}$. This result may then be compared with (7.17) to establish (7.15).

We begin by expanding $g(\xi)$ in powers of $(\xi - t)$. By the binomial theorem

$$\begin{aligned} g(\xi) = \xi^{\mu} &= [t + (\xi - t)]^{\mu} \\ &= t^{\mu}\left(1 + \frac{\xi - t}{t}\right)^{\mu} \\ &= t^{\mu} \sum_{k=0}^{\infty} \binom{\mu}{k} \left(\frac{\xi - t}{t}\right)^{k}. \end{aligned} \tag{7.18}$$

Considered as a power series in $(\xi - t)/t$, the radius of convergence is 1. Using Raabe's test we see that the series converges absolutely for

$$\frac{\xi - t}{t} = \pm 1.$$

Furthermore, it converges to ξ^{μ}. Since

$$\left| \binom{\mu}{k} \left(\frac{\xi - t}{t}\right)^{k} \right| \leqq \left| \binom{\mu}{k} \right|$$

for all $(\xi - t)/t$ in $[-1, 1]$, the Weierstrass M-test implies that the convergence is uniform in the closed interval $[-1, 1]$. Thus (7.18) converges uniformly for $\xi \in [0, t]$.

It therefore follows (see the note after Theorem 6, p. 75) that

$$D^{-\nu}[t^{\lambda} t^{\mu}] = \sum_{k=0}^{\infty} (-1)^{k} \frac{\Gamma(\nu + k)}{k!\,\Gamma(\nu)} [D^{k} t^{\lambda}][D^{-\nu-k} t^{\mu}] \tag{7.19}$$

is valid for $\nu > 0$, $t > 0$, $\lambda, \mu \geqq 0$. Thus

$$D^{-\nu}[t^{\lambda}t^{\mu}] = t^{\lambda+\mu+\nu}\frac{\Gamma(\mu+1)}{\Gamma(-\lambda)\Gamma(\nu)}\sum_{k=0}^{\infty}\frac{\Gamma(-\lambda+k)\Gamma(\nu+k)}{\Gamma(\mu+\nu+k+1)}\frac{1}{k!}$$

$$= t^{\lambda+\mu+\nu}\frac{\Gamma(\mu+1)}{\Gamma(\mu+\nu+1)}{}_2F_1(-\lambda,\nu,\mu+\nu+1;1).$$

If we equate this result to (7.17), we obtain

$$\frac{\Gamma(\lambda+\mu+1)}{\Gamma(\lambda+\mu+\nu+1)} = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\nu+1)}{}_2F_1(-\lambda,\nu,\mu+\nu+1;1). \tag{7.20}$$

In more conventional notation let $a = -\lambda$, $b = \nu$, $c = \mu + \nu + 1$. Then (7.20) becomes

$$\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} = {}_2F_1(a,b,c;1) \tag{7.21}$$

for

$$a \leqq 0, \qquad c - 1 \geqq b > 0. \tag{7.22}$$

Now (7.21) is the same as (7.15). And we know that this formula is valid for

$$c - a - b > 0 \tag{7.23}$$

with c unequal to a nonpositive integer. Thus we have established (7.21) only under the more restrictive conditions of (7.22). But we have encountered this phenomenon before (see pp. 71–73).