Lectures on Approximation Theory (Math 443)

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Chapter 1

General Approximation Problem

1.1 Normed Linear Spaces

There are several reasons for studying approximation theory and method ranging from a need to represent functions in computer calculations to an interest in the mathematics of the subject (replace a complicated function by one which is simpler and more manageable). Although approximation algorithms are used throughout the science and in many industrial and commercial fields, and to find a simple function which gives a best fit to the experimental data. The problem of approximating a given function or a table of values by a class of simpler functions has been of great interest theoretically and practically. For instance we may approximate the solution of a differential equations by a function of a certain form that depends on adjustable parameters. Here the measure of goodness of the approximation is a scaler quantity that is derived from the residual occurs when the approximating function is substituted into the differential equation (this scaler quantity is called a norm, which is a convenient measure of the "error" in the approximation).

Definition 1.1. *A non empty set* $\mathbb{F}$ *is called a linear space over a field of real numbers* $\mathbb{R}$ *if and only if for all* $A, B, C \in \mathbb{F}$ *and for all real numbers* r, s

(1) $A + B = B + A$.

(2) $A + (B + C) = (A + B) + C$.

(3) There is a unique element 0 in $\mathbb{F}$ such that $A + 0 = A \quad \forall A \in \mathbb{F}$.

(4) For each $A \in \mathbb{F}$ there is a unique element $-A \in \mathbb{F}$ such that $A + (-A) = 0$.

(5) $r \cdot (A + B) = r \cdot A + r \cdot B$.

(6) $(r + s) \cdot A = r \cdot A + s \cdot A$.

(7) $(r \cdot s) \cdot A = r \cdot (s \cdot A)$.

(8) $1 \cdot A = A$.

Definition 1.2. *Let $\mathbb{F}$ be a linear space and let $\|\cdot\| : \mathbb{F} \to \mathbb{R}$ such that*

(1) $\|A\| > 0$ unless $A = 0$.

(2) $\|rA\| = |r|\|A\|$ where r is scaler.

(3) $\|A + B\| \leqslant \|A\| + \|B\|$.

Then $\|\cdot\|$ defines a norm on $\mathbb{F}$.

Definition 1.3. *A linear space $\mathbb{F}$ equipped with a norm is called a normed linear space.*

Definition 1.4. *A metric space is a nonempty set M of points together with a function $d : M \times M \to \mathbb{R}$ satisfying the following properties for all x, y and $z \in M$*

(1) $d(x, y) = 0$ if $x = y$.

(2) $d(x, y) > 0$ if $x \neq y$.

(3) $d(x, y) = d(y, x)$.

(4) $d(x,z) \leqslant d(x,y) + d(y,z)$.

Remark 1.1. *In a normed linear space the formula $d(x,y) = \|x-y\|$ defines a metric. i.e., a normed linear space becomes a metric space.*

Proof. H.W. □

Definition 1.5.

$$C[a,b] = \{f : f : [a,b] \to \mathbb{R}, f \text{ is continuous}\}.$$

$$\mathbb{R}^N = \{(x_1, x_2, \ldots, x_N) : x_i \in \mathbb{R}, \text{for } i = 1, 2, \ldots, N\}.$$

The three norms that are used most frequently are the p-norms, for $p = 1, 2$ and ∞. For finite p the p-norm in $C[a,b]$ is defined as

$$\|f\|_p = \left[\int_a^b |f(x)|^p dx\right]^{\frac{1}{p}} \quad 1 \leqslant p < \infty$$

and the p-norm in $\mathbb{R}^N$ as

$$\|f\|_p = \left[\sum_{i=1}^{N} |f(x_i)|^p\right]^{\frac{1}{p}} \quad 1 \leqslant p < \infty$$

where $f = (f(x_1), f(x_2), \ldots, f(x_N))$. For $p = \infty$, the norms become,

$$\|f\|_\infty = \max_{a \leqslant x \leqslant b} |f(x)|$$

and

$$\|f\|_\infty = \max_{1 \leqslant i \leqslant N} |f(x_i)|$$

respectively.

The ∞-norm is called the Chebyshev norm (sometimes called the uniform or minimax norm).

Theorem 1.1. (Holder inequality) *If $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, then*

$$\int_a^b |A(x)B(x)|dx \leqslant \left[\int_a^b |A(x)|^p dx\right]^{\frac{1}{p}} \cdot \left[\int_a^b |B(x)|^q dx\right]^{\frac{1}{q}},$$

where $A, B \in C[a,b]$.