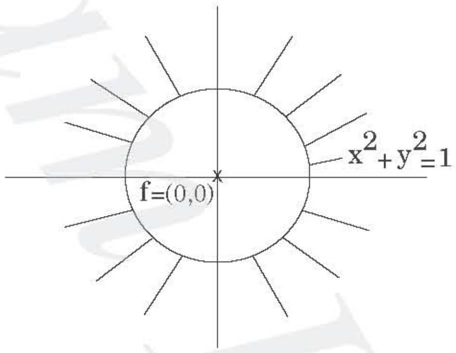
Example 1.4. *Let* $\mathbb{F} = \mathbb{R}^2$, $V = \{(x,y) : x^2 + y^2 > 1\}$, $f = (0,0)$, $\|\cdot\| = \|\cdot\|_2$. *Discuss existence of best approximation.*

Solution. The problem of determining the point in V which is nearest to $(0,0)$ has no solution. i.e., there is no best approximation to f from V.

If $V_1 = \{(x,y) : x^2 + y^2 \geqslant 1\}$ then all points that satisfy $x^2 + y^2 = 1$ is a best approximation to $f = (0,0)$. i.e., the best approximation to f from V_1 is exists and not unique.



Definition 1.7. *A sequence* $\{x_n\}$ *in a normed linear space is said to converge to a point* x^* *and we write* $x_n \to x^*$ *if* $\|x_n - x^*\| \to 0$ *as* $n \to \infty$.

Definition 1.8. *We say that* $\delta = \inf X$ *if there exists a sequence* $\{x_n\}_{n=1}^{\infty} \in X$ *such that* $x_n \to \delta$ *as* $n \to \infty$.

Definition 1.9. *An element* $h^* \in V$ *satisfying* $\|f - h^*\| = \inf_{h \in V} \|f - h\|$ *is called a best approximation of* f *with respect to* V.

Definition 1.10. *A subset* V *of* $\mathbb{F}$ *is said to be compact if every sequence of points in* V *has a subsequence which is converge to a point of* V.

Theorem 1.5. *Let* V *be a compact subset of* $\mathbb{F}$, *then there exists* $h^* \in V$ *such that*

$$\|f - h^*\| \leqslant \|f - h\| \quad \forall\, h \in V.$$

Proof. Let $\delta = \inf_{h \in V} \|f - h\|$. We want to show that there exists $h^* \in V$ such that $\|f - h^*\| = \delta$. From the definition of infimum there exists a sequence of points $\{h_n\}_{n=1}^{\infty} \in V$ such that $\|f - h_n\| \to \delta$ as $n \to \infty$. Since V is compact, it follows that there exists a subsequence of $\{h_n\}_{n=1}^{\infty}$ converging to $h^* \in V$.

$$f - h^* = (f - h_n) + (h_n - h^*).$$
$$\|f - h^*\| \leqslant \|f - h_n\| + \|h_n - h^*\|.$$

when $n \to \infty$ we get

$$\|f - h^*\| \leqslant \delta. \tag{1.5}$$

Note that

$$\delta = \inf_{h \in V} \|f - h\| \leqslant \|f - h\| \quad \forall\, h \in V.$$

Since $h^* \in V$ we get

$$\delta \leqslant \|f - h^*\|. \tag{1.6}$$

From (1.5) and (1.6) we get

$$\|f - h^*\| = \delta.$$

i.e.,

$$\|f - h^*\| \leqslant \|f - h\| \quad \forall\, h \in V.$$

$\therefore$ h^* is a best approximation of f. □

Remark 1.6. *Compactness of V is a sufficient condition for a best approximation to exist and not necessary.*

Example 1.5. *Let $\mathbb{F} = \mathbb{R}$, $V = (-\infty, 1]$, $\|\cdot\|_1 = |\cdot|$. Discuss existence of best approximation.*

Solution. *Note that V is not compact set, but there exists a best approximation to any point $f \in \mathbb{R}$.*

Theorem 1.6. *Let V be a finite dimensional subspace of a normed linear space $\mathbb{F}$, then there exists a best approximation in V to any point of $\mathbb{F}$.*

Proof. Let V be such a subspace and let $f \in \mathbb{F}$ be the prescribed point. Then if h_0 is an arbitrary point of V, the point sought lies in the set

$$\{h \in V : \|f - h\| \leqslant \|f - h_0\|\}.$$

This set is closed and bounded and thus compact, then by Theorem 1.5 there exists a best approximation in V to $f \in \mathbb{F}$. □

Remark 1.7. *It is not possible to drop the finite dimensional requirement of the above theorem.*

Example 1.6. *Let $\mathbb{F} = C[0, \frac{1}{2}]$ with the ∞-norm, $V =$ the space of polynomials of any degree.*

Solution. Let $f = \dfrac{1}{1-x}$, $h(x) = 1 + x + x^2 + \cdots + x^n \in V$.

$$\begin{aligned}\|f - h\| &= \max_{a \leqslant x \leqslant b} |f(x) - h(x)| \\ &= \max_{0 \leqslant x \leqslant \frac{1}{2}} |\frac{1}{1-x} - (1 + x + x^2 + \cdots + x^n)|.\end{aligned}$$

Thus any best approximation say h^* would satisfy $\|f - h^*\| = 0$ which implies $h^* = \dfrac{1}{1-x}$. This impossible and so no best approximation exists.

1.4 Uniqueness

We can investigate an example with regard to question (2).

Example 1.7. $\mathbb{F} = \mathbb{R}^2$, $V = \{(1, y) : y \in \mathbb{R}\}$, $f = (0, 0)$, $\|\cdot\| = \|\cdot\|_\infty$. *Discuss existence and uniqueness of best approximation.*

Solution. $\|f-h\|_\infty = \|(0,0)-(1,y)\| = \|(-1,-y)\|_\infty = \max\{1,|y|\}$

$$= \begin{cases} 1, & |y| \leqslant 1; \\ >1, & |y|>1. \end{cases}$$

$$|y| \leqslant 1 \quad \Rightarrow \quad -1 \leqslant y \leqslant 1.$$

Hence

$$\|f-h\|_\infty = \begin{cases} 1, & -1 \leqslant y \leqslant 1; \\ >1, & y<-1 \text{ or } y>1. \end{cases}$$

$\therefore$ any point $(1,y)$ such that $-1 \leqslant y \leqslant 1$ is a best approximation to $f=(0,0)$.

i.e., the best approximation to $f=(0,0)$ exists and not unique.

To discuss the uniqueness of best approximation we need to define a convex set.

Definition 1.11. *A set V of a linear space $\mathbb{F}$ is convex if $\forall\ x,y \in V$ implies that $\lambda x + (1-\lambda)y \in V$ for all $0 \leqslant \lambda \leqslant 1$.*

Geometrically: A set is convex if all line segments joining pairs of points in the set also belongs to the set.

Theorem 1.7. *If $f \in \mathbb{F}$ and V is a subspace of $\mathbb{F}$, then the set of best approximation to f from V, call it V^*, is convex.*

Proof. Let $h_1^*, h_2^* \in V^*$, we want to proof that $\lambda h_1^* + (1-\lambda)h_2^* \in V^*$.

$$h_1^* \in V^* \quad \Rightarrow \quad \|f-h_1^*\| \leqslant \|f-h\| \quad \forall\ h \in V.$$

$$h_2^* \in V^* \quad \Rightarrow \quad \|f-h_2^*\| \leqslant \|f-h\| \quad \forall\ h \in V.$$

$$\begin{aligned} \|f-(\lambda h_1^* + (1-\lambda)h_2^*)\| &= \|(\lambda f + (1-\lambda)f) - (\lambda h_1^* + (1-\lambda)h_2^*)\| \\ &= \|\lambda(f-h_1^*) + (1-\lambda)(f-h_2^*)\| \\ &\leqslant \lambda\|f-h_1^*\| + (1-\lambda)\|f-h_2^*\| \\ &\leqslant \lambda\|f-h\| + (1-\lambda)\|f-h\| \quad \forall\ h \in V \\ &\leqslant \|f-h\| \quad \forall\ h \in V. \end{aligned}$$

$\therefore$ $\lambda h_1^* + (1-\lambda)h_2^*$ is a best approximation to f.

Hence $\lambda h_1^* + (1-\lambda)h_2^* \in V^*$. □