Remark 1.8. *This theorem has a consequence that if there are two distinct best approximation to f there are infinitely many. i.e., the set of best approximation consists either on one element or infinitely many.*

Definition 1.12. *We say that $\mathbb{F}$ is a strictly convex normed linear space if $f_1 \neq f_2$, $\|f_1\| = r$, $\|f_2\| = r$, then $\|\lambda f_1 + (1-\lambda) f_2\| < r$ for all λ satisfying $0 < \lambda < 1$.*

Example 1.8. *Let $\mathbb{F} = C[0,1]$, $f_1(x) = 2x$, $f_2(x) = 3x^2$, $\|\cdot\| = \|\cdot\|_1$. Show that $\|\cdot\|_1$ is not strictly convex.*

Solution. Note that $f_1 \neq f_2$.

$$\|f_1\|_1 = \int_a^b |f_1(x)|dx = \int_0^1 |2x|dx = \int_0^1 2xdx = x^2\Big|_0^1 = 1.$$
$$\|f_2\|_1 = \int_a^b |f_2(x)|dx = \int_0^1 |3x^2|dx = \int_0^1 3x^2dx = x^3\Big|_0^1 = 1.$$

Take $\lambda = \frac{1}{2}$.

$$\|\lambda f_1 + (1-\lambda) f_2\|_1 = \|\frac{1}{2}(2x) + \frac{1}{2}(3x^2)\|_1 = \|x + \frac{3}{2}x^2\|_1 = \int_0^1 |x + \frac{3}{2}x^2|dx$$
$$= \int_0^1 (x + \frac{3}{2}x^2)dx = \frac{x^2}{2} + \frac{x^3}{2}\Big|_0^1 = 1.$$

Hence

$$\|\lambda f_1 + (1-\lambda) f_2\|_1 = 1.$$

$\therefore$ $\|\cdot\|_1$ is not strictly convex.

Theorem 1.8. *In a strictly convex normed linear space $\mathbb{F}$. A finite dimensional subspace V contains a unique best approximation to any point $f \in \mathbb{F}$.*

Proof. There exists a best approximation to f from V because V is a finite dimensional subspace.

Suppose that h_1 and h_2 are two distinct best approximation to f.

Take $\lambda = \frac{1}{2}$.

$\Rightarrow \quad \frac{1}{2}h_1 + \frac{1}{2}h_2$ is also a best approximation (by convexity).

suppose that

$$\|f - h_1\| = \|f - h_2\| = \|f - (\frac{1}{2}h_1 + \frac{1}{2}h_2)\| = r.$$

Put $f_1 = f - h_1$, $f_2 = f - h_2$ and notice that $f_1 \neq f_2$ and $\|f_1\| = r$, $\|f_2\| = r$.

Since $\mathbb{F}$ is strictly convex normed linear space

$$\|\frac{1}{2}f_1 + \frac{1}{2}f_2\| < r. \tag{1.7}$$

$$\|\frac{1}{2}f_1 + \frac{1}{2}f_2\| = \|\frac{1}{2}(f - h_1) + \frac{1}{2}(f - h_2)\| = \|f - (\frac{1}{2}h_1 + \frac{1}{2}h_2)\| = r.$$

$\therefore \quad \|\frac{1}{2}f_1 + \frac{1}{2}f_2\| = r.$

This contradicts inequality (1.7).

$\therefore \quad h_1 = h_2.$

i.e., the best approximation is unique. □

Remark 1.9. *If f and g are in $C[a,b]$ or $\mathbb{R}^N$, then*

$$\|af + bg\|_2^2 = a^2\|f\|_2^2 + 2ab < f, g > +b^2\|g\|_2^2,$$

where a, b are constants.

Proof.

$$\begin{aligned}\|af + bg\|_2^2 &= < af + bg, af + bg > \\ &= a^2 < f, f > +ab < f, g > +ba < g, f > +b^2 < g, g > \\ &= a^2\|f\|_2^2 + 2ab < f, g > +b^2\|g\|_2^2.\end{aligned}$$

□

Theorem 1.9. *The normed linear space $C[a,b]$ or $\mathbb{R}^N$ with the 2-norm is strictly convex.*

Proof. Let f and g be any two distinct points of $C[a,b]$ or $\mathbb{R}^N$ such that $\|f\|_2 = \|g\|_2 = r$ and note that

$$\|\lambda f + (1-\lambda)g\|_2^2 + \lambda(1-\lambda)\|f-g\|_2^2 = \lambda^2\|f\|_2^2 + 2\lambda(1-\lambda) < f,g > +(1-\lambda)^2\|g\|_2^2$$
$$+\lambda(1-\lambda)\{\|f\|_2^2 - 2 < f,g > +\|g\|_2^2\}$$
$$= \lambda^2 r^2 + (1-\lambda)^2 r^2 + 2\lambda(1-\lambda)r^2$$
$$= r^2\{\lambda^2 + (1-2\lambda+\lambda^2) + (2\lambda - 2\lambda^2)\}$$
$$= r^2.$$

$\therefore \quad \|\lambda f + (1-\lambda)g\|_2^2 + \lambda(1-\lambda)\|f-g\|_2^2 = r^2.$

$f \neq g \quad \Rightarrow \quad f-g \neq 0 \quad \Rightarrow \quad \lambda(1-\lambda)\|f-g\|_2^2 > 0.$

$\Rightarrow \quad \|\lambda f + (1-\lambda)g\|_2^2 < r^2.$

$\Rightarrow \quad \|\lambda f + (1-\lambda)g\|_2 < r.$

$\therefore$ the normed linear space $C[a,b]$ or $\mathbb{R}^N$ with the 2-norm is strictly convex. □

Remark 1.10.

(1) *From Theorem 1.8 and 1.9, we obtain that any point $f \in C[a,b]$ or $\mathbb{R}^N$ has a unique best approximation in the finite dimensional subspace $V \subset C[a,b]$ or $\mathbb{R}^N$ with the 2-norm.*

(2) *It has been stated already that the 1-norm and ∞-norm in $C[a,b]$ and in $\mathbb{R}^N$ are not strictly convex. If we prove that the norms are not strictly convex, then Theorem 1.8 does not answer the uniqueness question, but if we can demonstrate that a best approximation from a linear subspace is not unique , then we may deduce from Theorem 1.8 that the norm is not strictly convex.*

We give examples of this kind. In each one there is a linear subspace V and a point f such that the best approximation from V to f is not unique , where V and f contained in either $C[a,b]$ or in $\mathbb{R}^N$ and where the accuracy of the approximation is measured either by the 1-norm or by the ∞-norm.

Example 1.9. *Let $\mathbb{F} = C[-1,1]$, V=one dimensional linear space that contains all functions of the form $h(x) = \lambda x$, $-1 \leqslant x \leqslant 1, -1 \leqslant \lambda \leqslant 1$, $f(x) = 1$, $\|\cdot\| = \|\cdot\|_1$. Show that $\|\cdot\|_1$ is not strictly convex.*

Solution. $\|f-h\|_1 = \int_a^b |f(x)-h(x)|dx = \int_{-1}^{1} |1-\lambda x|dx.$

$$-1 \leqslant x \leqslant 1,\ -1 \leqslant \lambda \leqslant 1 \quad \Rightarrow \quad -1 \leqslant -\lambda x \leqslant 1 \quad \Rightarrow \quad 0 \leqslant 1-\lambda x \leqslant 2.$$

Hence

$$\|f-h\|_1 = \int_{-1}^{1} (1-\lambda x)dx = x - \frac{\lambda}{2}x^2\Big|_{-1}^{1} = (1-\frac{\lambda}{2}) - (-1-\frac{\lambda}{2}) = 2.$$

$\therefore \quad \|f-h\|_1 = 2 \quad \forall\, h \in V. \quad \Rightarrow \quad \min_{h\in V} \|f-h\|_1 = 2.$

Hence the best approximation is not unique. i.e., the 1-norm in $C[-1,1]$ is not strictly convex.

Example 1.10. *Let* $\mathbb{F} = \mathbb{R}^N$, $V = \{h : h = (\lambda x_1, \lambda x_2, \ldots, \lambda x_N), -1 \leqslant x_i \leqslant 1$ *for all* $i = 1,2,\ldots,N, -1 \leqslant \lambda \leqslant 1\}$, $f(x) = (1,1,\ldots,1) \in \mathbb{R}^N$, $\|\cdot\| = \|\cdot\|_1$. *Show that* $\|\cdot\|_1$ *is not strictly convex.*

Solution. divide the interval $[-1,1]$ by the points $-1 = x_1 < x < \cdots < x_N = 1$ which are equally spaced.

$$h = \frac{b-a}{N-1} = \frac{1-(-1)}{N-1} = \frac{2}{N-1}.$$

$\therefore \quad x_{i+1} - x_i = \dfrac{2}{N-1}.$

$$x_i = x_1 + (i-1)h = -1 + \frac{2(i-1)}{N-1} \quad i = 1,2,\ldots,N.$$

$$\|f-h\|_1 = \sum_{i=1}^{N} |f(x_i) - h(x_i)| = \sum_{i=1}^{N} |1-\lambda x_i|.$$

$$-1 \leqslant x_i \leqslant 1,\ -1 \leqslant \lambda \leqslant 1 \quad \Rightarrow \quad -1 \leqslant -\lambda x_i \leqslant 1 \quad \Rightarrow \quad 0 \leqslant 1-\lambda x_i \leqslant 2$$

Hence

$$\|f-h\|_1 = \sum_{i=1}^{N}(1-\lambda x_i) = \sum_{i=1}^{N} 1 - \lambda \sum_{i=1}^{N} x_i = N - \lambda \sum_{i=1}^{N} x_i.$$

$$\sum_{i=1}^{N} x_i = \sum_{i=1}^{N}\Big[-1+\frac{2}{N-1}(i-1)\Big] = -N + \frac{2}{N-1}(1+2+\cdots+N-1) = -N+N = 0.$$

$\therefore \quad \|f-h\|_1 = N \quad \forall\, h \in V. \quad \Rightarrow \quad \min_{h\in V} \|f-h\|_1 = N.$

Hence the best approximation is not unique. i.e., the 1-norm in $\mathbb{R}^N$ is not strictly convex.